Synopsis Report: Topological Insulators

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1 Introduction

1.1 What are Topological Insulators

Topoological Insulators are a newly discovered phase of matter that have become one of the hottest topics in condensed-matter physics. The first "electronic phases" of matter to be defined were the electrical conductor and insulator, and then came the semiconductor, the magnet and more exotic phases such as the superconductor. Topological Insulators are the newest addition to this family of electronic phases of matter.

Topological Insulators can insulate on the inside but conduct on the outside acting like a thick plastic cable covered with a layer of metal, except that the material is actually the same throughout. Moreover, the conducting electrons arrange themselves into spin-up electrons traveling in one direction, and spin-down electrons traveling in the other; this "spin current" is a milestone in the realization of practical "spintronics". They are robust and unlike other materials their properties don't change with imperfections on the surface or impurities which make them a very useful material and can have many novel applications.

Although topological phenomenon as the quantum Hall effect had already been found in 2D ribbons in the early 1980s, but the discovery of the first example of a 3D topological phase in Topological insulators reignited that earlier interest. The topological insulator states in 2D and 3D materials were predicted theoretically in 2005 and 2007, prior to their experimental discovery in 2007 so they are a new and exciting field.

1.2 How do the work

Unlike superconductors and magnets, which have order associated with a broken symmetry, topologically ordered states are distinguished by a kind of knotting of the quantum states of the electrons. The surface states of topological insulators are topologically protected, which means that unlike ordinary surface states they cannot be destroyed by impurities or imperfections. So topology plays an important role.

While topology can be a quite abstruse branch of mathematics, some of its concepts are familiar to anyone who has tied a knot. Consider the linked rings in the Olympic symbol, for example. Without cutting a ring it is impossible to separate them, even if the rings are bent, enlarged or shrunk. The "linking number" that formalizes this idea is an example of a topological invariant, which is a quantity that does not change under continuous changes of the rings. Topological ideas of this type were first applied to quantum condensed-matter physics in the 1980s to understand the integer quantum Hall effect. The effect of a magnetic field in this phase, which breaks timereversal symmetry, is to "knot" the electronic wavefunction in a non-trivial way - the wavefunctions in a quantum-Hall sample cannot be smoothly changed, with the system remaining insulating, to those of an ordinary insulator or vacuum. As a result, a metallic layer appears at the surface where the wavefunction topology changes, and the properties of this layer are not very sensitive to exactly how the surface is made.

1.3 Some Applications

One of the exciting potential applications of topological insulators is the creation of Majorana fermions. These elusive fundamental particles have been discussed in particle physics for decades, though as yet there has been no definitive proof of their existence. It is being proposed that a combination of a Topological Insulator and a Superconductor could be used to find this elusive particle.

They could be used for fault tolerant quantum computation, currently one of the problems with practically realizing quantum computers is the robustness of the quantum systems, they are so sensitive that even a small fluctuation of even one electron which could be displaced because a truck was moving a mile away collapses the computation.

The field of electronics called 'Spintronics'. Previously it was hard to separate the spins and Stern Gerlach experiment was used to separate them using a magnetic field but in Topological Insulators, the spins are separated and they flow as separate currents in separate directions.

2 Thesis Outline

It would be a year long project and we have divided the thesis in to five chapters. Following is the general outline

2.1 Chapter 1

The Adiabatic approximation, Berry phases, Relation to the Aharanov Bohm Effect, relation to the magnetic monopoles.

2.2 Chapter 2

Solve single spin 1/2 particles in a magnetic field and calculate the Berry phase, do it for spin 1 particles (3x3 matrices).

2.3 Chapter 3

Understanding Fractional Quantum Hall Effect from the point of view of Berry Phases, this is the Hamiltonian approach (paper by Shankar).

2.4 Chapter 4

Topoloogy and Condensed Matter Physics.

2.5 Chapter 5

Understanding Topological Insulators from the point of view of Berry phases and forms.

3 What I have done so far

3.1 The Adiabatic Theorem and Born Oppenheimer approximation

Imagine a pendulum, with no friction or air resistance oscillating in a box. If you grab the support and shake it in a jerky manner the bob will swing around chaotically. But if you very gently and steadily move the support, the pendulum will continue to oscillate in a nice way. This gradual change of the external conditions is known as an adiabatic process.

We use this adiabatic processes for Born Oppenheimer approximations, we start with nuclei at rest and solve for the motion of the electron and calculate the electronic wavefunctions, and using these to obtain the positions and relatively sluggish motion of the nuclei.
In quantum mechanics this approximation can be cast in to a theorem, if a particle is initially in the nth eigenstate of H^i. it will be carried (after an adiabatic process) to the nth eigenstate of H^f with a phase factor. For example if you have a particle in a infinite potential well, and we slowly increase the width of the well from a to 2a, if we started from a particle in the ground state, it would still remain in the ground state but probably with a phase factor. Note that this is not a small change (like in perturbation theory) but a huge change. But if we had expanded the width of the tunnel suddenly, the results won't have been the same and we would have ended up with a complicated linear combination of wave functions.

3.2 Berry phases

So a very nice proof of the adiabatic theorem is given in both Shankar and Griffiths, I won't dwell on it here but would just give the results. So a particle starting in the nth eigenstate ψ_n then after going through a adiabatic process it picks up a couple of phases and we have:

$$\psi_n(t) = e^{i\theta_n(t)}\ e^{i\gamma_n(t)}\ \psi_n(t) \tag{1}$$

Where $\gamma_n(t)$ is a geometric phase, and $\theta_n(t)$ is a dynamical phase. Where

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t)dt \tag{2}$$

$$\gamma_n(t) = i\int_0^t < \psi_n(t)|\frac{\partial}{\partial t}\psi_n(t) > dt \tag{3}$$

Now $\psi_n(t)$ depends on t because there is some parameter R(t) in hamiltonian that is changing with time (like the width of the square well we talked about before). So we can write our phase change in the form of the parameters as

$$\gamma_n(T) = i\oint < \psi_n|\nabla_R\psi_n > dR \tag{4}$$

This is a line integral around a closed loop and is not in general zero. It was first obtained by Berry in his 1984 paper and this $\gamma_n(T)$ is known as Berry phase. Notice that the Berry phase only depends on the path taken. However the dynamical phase depends on the elapsed time. It was beleived before that the geometric phase factor was of no physical significance and it's arbitrary, but Berry showed that if you carry a hamiltonian around a close loop the relative phase at the end and beginning can be measured.

We then calculate the Berry phases for a spin 1/2 particle in a magnetic field but I would not derive it here. But the result is actually interesting and delightfully simple. $\gamma_+(T) = -\frac{1}{2}\Omega$ where Ω is the solid angle. It says that if you take a magnet and lead the electron's spin around adiabatically in an arbitrary closed path, the net geometric phase will be minus one half of the solid angle subtended by the magnetic field vector.

3.3 Berry Connections and Berry Curvature

We know how magnetic flux is written in terms of a vector potential:

$$\Phi = \int_S B.da \tag{5}$$

As $B = \nabla \times A$ and we can apply stokes theorem:

$$\Phi = \int_S (\nabla \times A).da = \oint_C A.dr \tag{6}$$

This expression is similar to equation 4, The Berry's phase can be thought of as the 'flux' of a 'magnetic field'. We define this Vector potential like quantity as Berry's Connection.

$$A_n(R) = i < n(R)|\nabla_R|n(R) > \tag{7}$$

And the magnetic field like expression as a Berry curvature.

$$'B' = \nabla_R \times A_n(R) \tag{8}$$

And Berry Phase is just the line integral of the Berry's connection.

$$\gamma_n(T) = i \oint A_n(R).dR \tag{9}$$

Berry connection or Berry potential is gauge-dependent, transforming as $A_n(R) = A_n(R) + \nabla_R \beta(R)$. Hence the local Berry connection can never be physically observable. However, its integral along a closed path, the Berry phase γ_n, is gauge-invariant up to an integer multiple of 2π. Thus, is absolutely gauge-invariant, and may be related to physical observables. In contrast to the Berry connection, which is physical only after integrating around a closed path, the Berry curvature is a gauge-invariant local manifestation of the geometric properties of the wavefunctions in the parameter space, and has proven to be an essential physical ingredient for understanding a variety of electronic properties.

There is a nice example given in which we have spinors in a magnetic field, we calculate the relevant Berry's connection and Berry Curvature for it. Then we gauge transform the spinors (multiply with a phase factor), and after computing again the Berry's connection changes but the Berry Curvature remains the same. This is consistent with the conclusion that the Berry connection is gauge-dependent while the Berry curvature is not.

3.4 The Aharanov Bohm Effect and Berry Phases

A detailed explanation of the phenomenon is given in the other notes I wrote on Latex from the lecture by Dr. Hoodbhoy, so I won't go over it in detail over here. But I would discuss the main points. So essentially we have a thin solenoid which has a B field going through it and there is a flux associated with it. We take a beam of electrons, we split it and go around this solenoid and combine the beam. So we would expect that the electrons would not experience any change and won't notice the flux tube as essentially they don't experience the magnetic field and there is no lorentz force but what is startling is that when we combine the split beams there is a phase difference, which dependent on B field and also a difference in energies which has a B field term.

So Berry in his famous paper of 1984 showed that Berry phases neatly confirms the AB effect and so it's a particular instance of geometric phases. And by using the Berry's phase formula we can calculate the phase difference.

3.5 Berry Phases and Magnetic Monopoles

Now we know that there is a Berry vector potential which we have been calling Berry's connection. So normally when we have a vector potential, we take it's curl and get a corresponding magnetic field. And the source of the magnetic field is either a current or a magnetic monopole if they exist. So the question is what is producing the Berry potential, so it is a magnetic monopole but in parameter space. The introduction in the paper by Jackiw explains this point beautifully, so like in Born Oppenheimer approximation when we initially fix the slow parameters and then first analyse for the fast variables. An external vector potential is induced in the parameter space by these fast variables and that comes into play when we solve for the slow variables.

I would just give an overview here without the mathematics and derivation. So we first define our parameter space by coordinates $\mathbf{R} = (R, \theta, \phi)$, we consider a hamiltonian and we write a spinor for it :

$$|+, \theta, \phi\rangle = \begin{pmatrix} \cos(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2})e^{i\phi} \end{pmatrix} \tag{10}$$

Now as it does not depend on R we take a unit sphere and analyse. Observe that the lower component does not approach a unique value as we approach the south pole. Though this is not the problem at the north pole as there the lower component is just zero. So thus we really have not defined the spinor globally. So what if we multiply the whole spinor with a $e^{-i\phi}$, we now have a spinor well defined near the south pole and not at the north pole. So it follows that we can only define the spinors in patches in the parameter space, for this problem two patches would do.

By Berry's potential formula we know that this problem would still persist as it is just the derivative of the spinor so it follows that berry potential is also not defined globally and is defined in patches.

A global vector potential is not possible in the presence of a magnetic monopole. take a loop on the latitude of a sphere near the north pole, by stokes theorem the line integral would be equal to the flux coming out, (flux due to the monopole at the centre of the sphere) but as you keep on increasing the loop and move past the equator towards the south pole the line integral monotonically increases but it doesn't vanish at the south pole as expected. Infact there is a singularity there which is equal to the full monopole flux. The berry vector potential is given by:

$$A = -\frac{\hbar}{2} e_{\phi} \frac{1 - cos\theta}{Rsin\theta} \tag{11}$$

Observe the singularity at the south pole, this is called the dirac string.The vector potential does not describe a monopole at the origin, but one where a tiny tube (the dirac string) comes up the negative z axis, smuggling in the entire flux to the origin from which it emanates radially. The string flux is the reason the tiny loop around the south pole gives a non zero answer equal to the total flux.

As it is spherically symmetric, we can move around the dirac string anywhere on the sphere with a gauge transformation. This is also like the AB effect as the infinite solenoid of flux is like a dirac string. In shankar it does a nice treatment and knits together the two patches, with a different vector potential in each demanding they overlap at a place like the equator. The two potentials differ by a single valued gauge transformation and you can recover the dirac quantisation condition from it.

You can also get this result by Holonomy, a nice derivation is given in chapter 3 of Wilcek.

3.6 Abelian and Non Abelian Gauge theories

Using the book by Kane as reference I understood the different gauge theories, relativistic notations, covariant derivatives and looked at Berry potential for non abelian cases, went through the paper by Jackiw and saw Berry's curvature for both abelian and non abelian cases. This paper shows how symmetries and conservation laws are effected by Berry's Connection. When we solve for the fast variables we get a $H_{effective}$, and if constants of motion commuted with H then they would also commute with the effective hamiltonian, but these new commuting constants of motions would also have to be modified . For example the rotational symmetry is not lost and angular momentum also commutes with the effective hamiltonian but is modified by the berry connection. The angular momentum that commutes with $H_{effective}$ has another term in addition to the kinematical angular momentum.

4 References

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