The Rotating Black Hole

Syed Ali Raza

May 24, 2012

1 Abstract

In this project we explore some of the properties of space time near a spinning black hole. Analogous properties describe spacetime near other stellar spinning objects like Earth and the Sun too. Near a spinning black hole you are swept along tangentially in the direction of rotation, we would discuss the various properties of spacetime by analysing the Kerr metric that describes the rotating black hole. We would use the test cases of extreme spin and static limit to get an intuitive feel of the physics going on in describing the phenomenon. We would use light as a test to see how the tangential and radial velocities of particles vary. We would discuss negative energy and how the penrose process can be used to extract energy from a rotating black hole.

This project is basically one of the projects suggested on the MIT OCW website as part of their GR course. I followed the project F in the book,'Exploring Black Holes' by Wheeler and Taylor which is a really nice book on guides to projects in GR. It gives a basic overview and then makes you derive equations and answer many queries along the way. I have tried to answer most of the queries as part of this project.

2 The Kerr Metric in the Equatorial Plane

We explained stationary black holes with the Schwarzschild metric, for rotating black holes we have the Kerr metric, we have expressed them here in what are known as Boyer-Lindquist coordinates. The angular momentum of the black hole is represented by J. We define a angular momentum parameter 'a', where $a = \frac{J}{M}$, M is the mass of the black hole.

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{4Ma}{r} dt d\phi - \frac{dr^2}{1 - \frac{2M}{r} + \frac{a^2}{r^2}} - \left(1 + \frac{a^2}{r^2} + \frac{2Ma^2}{r^3}\right) r^2 d\phi^2 \tag{1}$$

This is the Kerr Metric, we have written it in terms of the Boyer-Lindquist coordinates which in terms of cartesian coordinates are:

$$x = \sqrt{r^2 + a^2} sin\theta cos\phi \tag{2}$$

$$y = \sqrt{r^2 + a^2} sin\theta sin\phi \tag{3}$$

$$z = r cos\theta \tag{4}$$

Note that the metric only holds in the equatorial plain perpendicular to the axis of rotation. The Kerr metric provides a complete description of the space time in the equatorial plane of the rotating black hole. The Kerr metric in the limit of zero angular momentum simply reduces to the Schwarzschild metric.

The first new feature of the Kerr metric compared to the Schwarzschild metric is the coefficient of dr^2. For Schwarzschild it was $1/(1 - 2M/r)$, but for Kerr it is also dependent on the angular momentum parameter a. The point of no return or the Schwarzschild radius in a stationary black hole is $r_H = 2M$, this is the point where we have a singularity and the coefficient of dr^2 blows up. For the rotating black hole we can find the radius of horizon by setting the denominator in the dr^2 term equal to zero. And we get:

$$r_H = M \pm (M^2 - a^2)^{1/2} \tag{5}$$

As you can see we get two radii of horizons and we would discuss what they represent in a while. Also by plugging in $a = 0$, you can check that one of the solution reduces to what we had for the stationary black hole.

3 The Kerr Metric for extreme angular momentum

To get an intuitive understanding and to not go into too much mathematics we will consider the case of extreme angular momentum. From equation 5 you can see, to have a real value of r_H, the maximum value is $a = M$. For this case the angular momentum is $J = M^2$. Also note that in this limit case you have only one r_H and both the inner and outer horizons have merged. A black hole spinning at this maximum rate is called an extreme Kerr black hole.

Now we will see how the Kerr metric changes in the extreme angular momentum limit. By setting $a = M$ in equation 1 you get:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \frac{4M^2}{r} dt d\phi - \frac{dr^2}{\left(1 - \frac{M}{r}\right)^2} - R^2 d\phi^2 \tag{6}$$

This equation has been simplified by defining:

$$R^2 = r^2 + M^2 + \frac{2M^3}{r} \tag{7}$$

Here R is the reduced circumference for extreme Kerr spacetime. And so we can find a corresponding r for this.

The second new feature of the Kerr metric, is the presence of the product $dtd\phi$ which is also known as a cross product. This term is responsible for frame dragging, one might say that the spacetime is swept around by the rotating black hole. You may think that if there is a rocket stuck in this rotating frame, it can use it's boosters in the tangential direction to keep itself from being swept away and be stationary, but turns out this is not the case. We would discuss it later in detail.

4 Static Limit

The third new feature of the Kerr metric is the presence of static limit. This is the limit in which the coefficient of the dt^2 term, $(1 - 2M/r)$ goes to zero in the Kerr metric. The static limit is $r_S = 2M$, note that it is independent of the angular momentum parameter a.

The static limit gets its name from the prediction that for radii smaller than r_S, but greater than that of the horizon r_H, the observer cannot remain at rest or be static. The space between the the static limit and the horizon limit is known as the ergosphere, $r_S > r > r_H$. Inside the ergosphere you are always dragged along the direction of rotation and not even tangential rockets would help you in staying stationary.

Now to get a better understanding of what happens in a ergosphere we will see how light travels in different ways in it. We would first consider a flash of light moving in the ϕ direction (dr = 0). This is just the initial push and later light can travel in any direction. Because this is light, the proper time is zero between adjacent events on the path. So we substitute $d\tau = 0$ in our metric equation and divide by dt^2 and we get:

$$R^2 \left(\frac{d\phi}{dt}\right)^2 - \frac{4M^2}{r}\left(\frac{d\phi}{dt}\right) - \left(1 - \frac{2M}{r}\right) = 0 \tag{8}$$

This is a quadratic in the angular velocity $\frac{d\phi}{dt}$, we solve and get the following:

$$\frac{d\phi}{dt} = \frac{2M^2}{rR^2} \pm \frac{2M^2}{rR^2}\left[1 + \frac{r^2R^2}{4M^4}\left(1 - \frac{2M}{r}\right)\right]^{1/2} \tag{9}$$

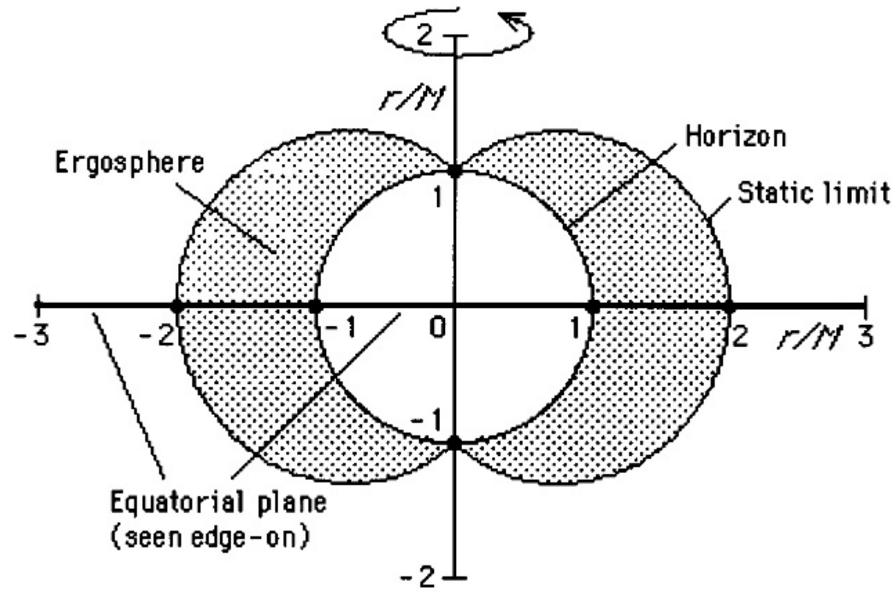

Figure 1: cross section of an extreme black hole showing the static limit and horizon. Between the horizon and static lies the ergosphere.

This expression at the static limit, $r_S = 2M$, has two solutions:

$$\frac{d\phi}{dt} = 0 \tag{10}$$

$$\frac{d\phi}{dt} = \frac{4M^2}{rR^2} = \frac{1}{3M} \tag{11}$$

The second solution represents light sent off in the same direction as the hole is rotating. The other solution says that the light sent in the other direction doesn't move at all as seen by the observer far away. The dragging of the hole has become so strong that even light can not move in the opposite direction. Hence any material particle, which must move slower than light will have to rotate with the hole, even if it has an arbitrarily large angular momentum in the opposite direction.

We have shown this property at just the static limit, $r_S = 2M$, we can similarly show that inside the ergosphere ($r_H < r < r_S$) in either of the tangential direction light would be dragged along the direction of the black hole.

The fourth new feature of the Kerr metric is the available energy. No net energy (except Hawking Radiation) can be extracted from the non spinning black hole. However energy is available from a spinning black hole and we would discuss it in detail later.

5 Radial and Tangential motion in light

First we consider the radial motion of light. For light ($d\tau = 0$) and moving in the radial direction ($d\phi = 0$). Substituting these are Kerr metric becomes:

$$\frac{dr}{dt} = \pm \left(1 - \frac{M}{r}\right) \left(1 - \frac{2M}{r}\right)^{1/2} \tag{12}$$

You can see that at the static limit, $r_S = 2M$, the radial speed of light goes to zero and is imaginary inside the ergosphere. Which means no real radial motion is possible inside the ergosphere.

Similarly we can derive an expression for the tangential velocity:

$$R\frac{d\phi}{dt} = \frac{2M^2}{rR} \pm \frac{r - M}{R} \tag{13}$$

We look at the light dragging at the horizon, r_H given by equation 5. By setting $a = M$, and plugging r_H back in the above expression we find that the initial tangential velocity at the horizon for light has a single value given by $R\frac{d\phi}{dt} = \frac{1}{2M}$.

Below are plots for the radial and tangential velocities of light along with the labeling of the ergosphere.

6 Comparison of results of non spinning and extreme-spin black holes

Below is a table that compares the non spinning and extreme-spin black holes, we have also derived the energy and angular momentum as constants of motion here by using the principle of extremal aging and other methods. However I won't go into its derivation at the moment.

7 Plunging

Near the non rotating black hole the simplest motion was a radial plunge, but what about in the spinning black hole's case? By setting the angular momentum term in the table equal to zero, we get the following expression:

$$\frac{d\phi}{dt} = \frac{2M^2}{rR^2} \tag{14}$$

This shows that even for a particle with zero angular momentum, when it comes near a spinning black hole, it circulates around it. Let's see if we can calculate the trajectory of such a particle too. By setting $E/m = 1$ for a stone with zero angular momentum we get the following:

$$\frac{E}{m} = 1 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2M^2}{r}\frac{d\phi}{d\tau} \tag{15}$$

Using the above two equations and doing some algebra we get the following relation:

$$dr = \frac{(r-M)^2}{r}\left[\frac{r^3}{2M^3} + \frac{r}{2M}\right]^{1/2} d\phi \tag{16}$$

We can simulate this and figure shows the particle's trajectory. For an observer faraway the stone spirals around the black hole then settles in a tight circular path at $r = M$. As you can see if you plug in $r = M$ into the above

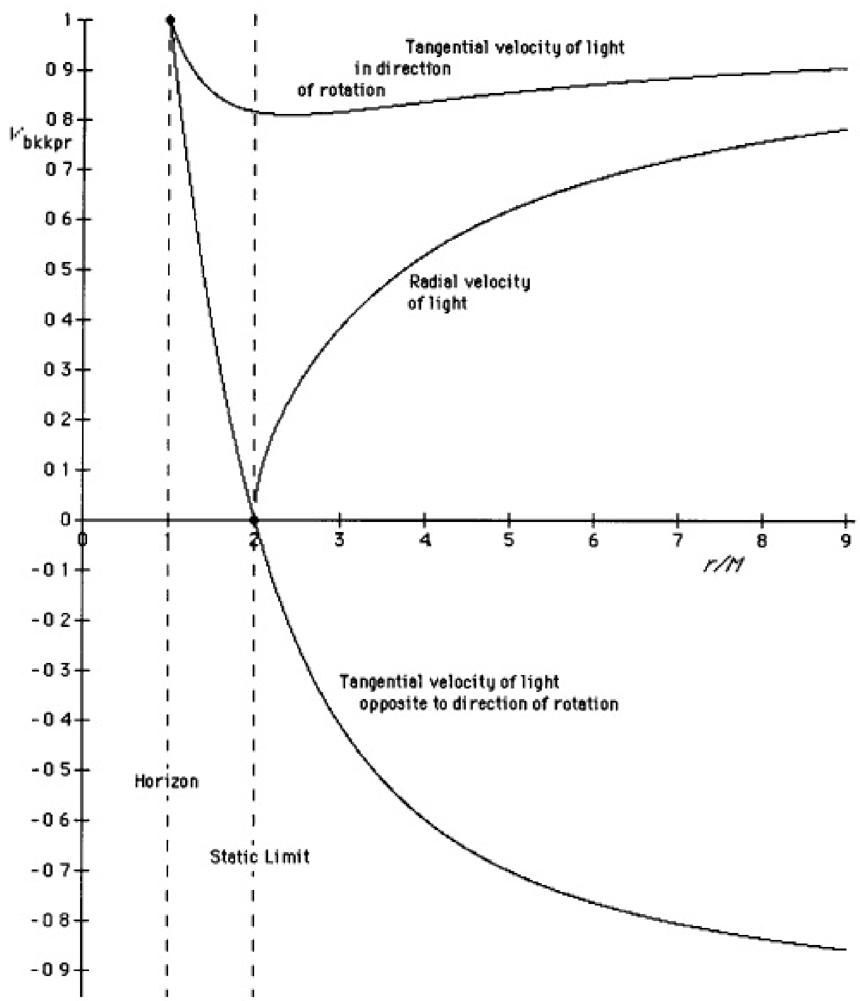

Figure 2: plot of radial and tangential velocities of light near an extreme Kerr black hole. Note all velocities approach minus or plus unity

Quantity	Nonspinning Schwarzschild black hole	Extreme-spin Kerr black hole ("shell" = stationary ring outside static limit)
Define r and R	Reduced circumference = $r \equiv \frac{\text{(circumference of shell)}}{2\pi}$	Reduced circumference R given by: $R^2 \equiv r^2 + M^2 + \frac{2M^3}{r}$
Shell time vs. far-away time: (gravitational red shift)	$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt$	$dt_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} dt$
dr_{shell} vs. dr	$dr_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{-1/2} dr$	$dr_{\text{shell}} = \left(1 - \frac{M}{r}\right)^{-1} dr$
Energy (constant of the motion)	$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau}$	$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2M^2}{r}\frac{d\phi}{d\tau}$
Angular momentum (constant of the motion)	$\frac{L}{m} = r^2\frac{d\phi}{d\tau}$	$\frac{L}{m} = R^2\frac{d\phi}{d\tau} - \frac{2M^2}{r}\frac{dt}{d\tau}$

Figure 3: A table of comparison of results of non-spinning and extreme-spin black holes.

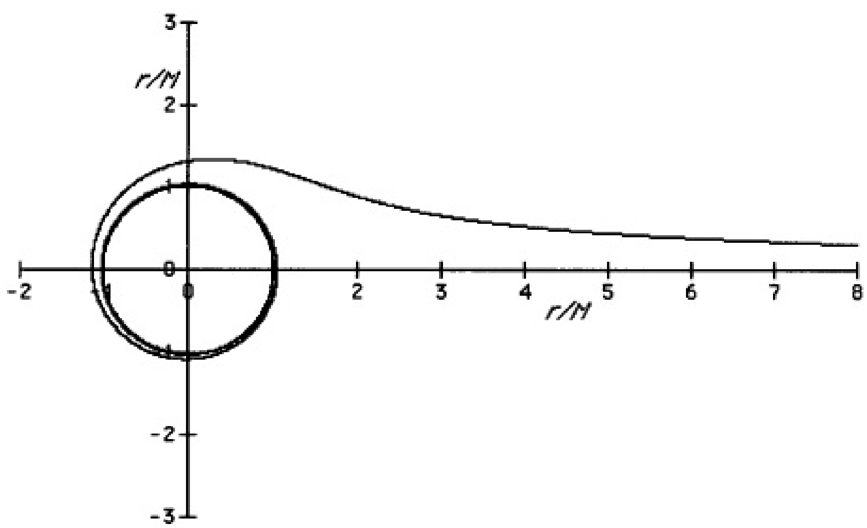

Figure 4: Kerr map of the trajectory in space of a stone dropped from rest far from a black hole. The stone spirals in to the horizon at r=M and circulates there forever.

equation $dr = 0$, so fixed radial motion.

Similarly we can find a relation between dr and $d\tau$ too:

$$\frac{dr}{dt} = \frac{2M}{r}\left(1 + \frac{M^2}{2r^2}\right) \tag{17}$$

You would see that in the limit of small radii, the velocity exceeds unity similarly to what happens in Schwarzschild too. This is absurd and we know that the Kerr Metric does not explain at the center of the black hole.

8 Negative Energy: The Penrose process

In the process, a lump of matter enters into the ergosphere of the black hole, and once it enters the ergosphere, it is split into two. The momentum of the two pieces of matter can be arranged so that one piece escapes to infinity, whilst the other falls past the outer event horizon into the hole. The escaping piece of matter can possibly have greater mass-energy than the original infalling piece of matter, whereas the infalling piece has negative mass-energy. In summary, the process results in a decrease in the angular momentum of the black hole, and that reduction corresponds to a transference of energy whereby the momentum lost is converted to energy extracted.

A consequence of these laws is that if the process is performed repeatedly, the black hole can eventually lose all of its angular momentum, becoming non-rotating, i.e. a Schwarzschild black hole.

But what is negative mass energy? For schwarzschild geometry the physical system differs from the newtonian system. A particle at rest near the horizon of a non spinning black hole has zero total energy. Which means that it takes an energy equal to its rest mass energy (m) to remove this particle at rest and take it to a large distance away from the black hole. However, in Kerr geometry the physical system differs from this Schwarzschild geometry. A particle can have a negative energy near a spinning black hole which means an energy greater than its rest energy (greater than m) is required to take such a particle to rest at a great distance away from the spinning black hole. This process would cause a decrease in the black hole's mass and angular momentum.

This entire strategy rests on the assumption that an object can achieve a state of negative energy. So we look at the expression of energy and see that indeed the energy can be negative:

$$\frac{E}{m} = 1 = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} + \frac{2M^2}{r}\frac{d\phi}{d\tau} \tag{18}$$

We would start by setting the energy equal to zero and get the following expression for the rate of change of angle:

$$\frac{d\phi}{dt} = \frac{2M - r}{2M^2} \tag{19}$$

You can see for $r > 2M$ the angular velocity is negative, and for $r < 2M$ the angular velocity is positive. We can see that the tangential velocity then becomes:

$$v_{E=0} = R\frac{d\phi}{dt} = \frac{R(2M - r)}{2M^2} \tag{20}$$

and by plugging it back in and doing some algebra we can show that the condition for the particle to have negative energy as following:

$$v_{E=neg} = R\frac{d\phi}{dt} = \frac{R(2M - r)}{2M^2} \tag{21}$$

Figure 5 gives a nice description of this.

9 A Practical Penrose process

Here we discuss a thought experiment of how an advanced civilisation could actually milk off energy from rotating black holes. Equal quantities of matter and anti matter are carried down to a rotating Kerr black hole. There the matter and anti matter are annihilated to form two oppositely moving beams of electromagnetic radiation. One pulse has negative energy and drops into the black hole, robbing the black hole of some of its mass and angular momentum. The other pulse has positive energy and escapes to a distant observer who can use this energy for practical purposes.

10 Refernces

- Taylor, Wheeler, Exploring Black Holes 2000.
- Misner, Thorne, and Wheeler, Gravitation, Freeman and Company, 1973.
- Energetics of the Kerr-Newman Black Hole by the Penrose Process; Manjiri Bhat, Sanjeev Dhurandhar Naresh Dadhich; J. Astrophys. Astr. (1985) - www.ias.ac.in.

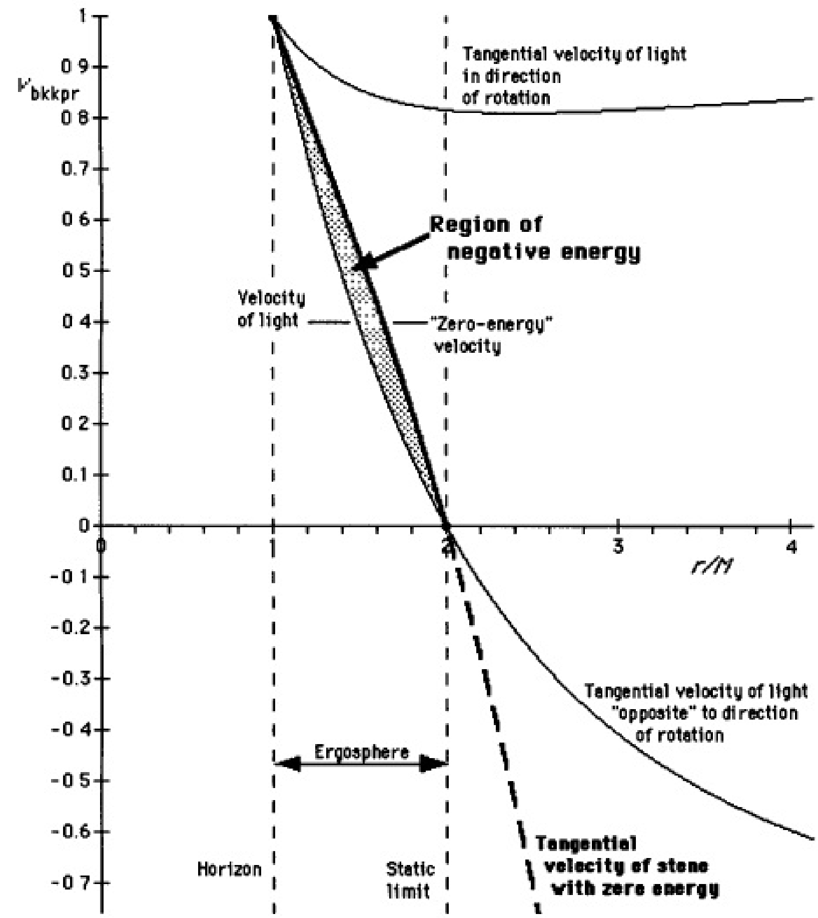

Figure 5: Tangential velocity of a stone with zero energy(thick curve). For r greater than the static limit 2M, the particle can not have zero or negative energy because it would have to be moving in a negative tangential direction with a speed greater than that of light in that direction. Only inside the ergosphere is the critical tangential velocity possible. The shaded area shows the negative energy region.