Quantum Hall Effect in a Spinning Disk Geometry

Syed Ali Raza

12100124

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Supervisor: Professor Pervez Hoodbhoy

Department of Physics

LUMS, School of Science and Engineering

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1 Introduction

We have shown that by spinning a Quantum Hall system you get a really nice quantized result. Another energy quantization in addition to the Landau quantization is observed and the degeneracy of the states in the Landau levels is lifted. The spacing between these levels is dependent on the frequency of spinning the disk and we have shown by calculations that you don't need to spin it at ridiculously high speeds to observe it. In fact, it can easily be done in the lab. This splitting causes an overall broadening of Landau levels and we can use it to mimic the broadening, that are due to impurities in the sample. The impurities are essential in quantization of conductance and there role is still not properly understood, but now we can control the broadening and overlap of peaks through the spinning frequency and can better understand the dependence. We verify this result by solving it through both the series solution approach and the operator approach.

After we got this interesting energy expression we wanted to see if spinning the Quantum Hall disk sample also changes the quantized conductance as it changes the quantized Energy spectrum. We first understood how to get the Kubo's formula of conductance from perturbation theory and how we can relate it to Topology and Chern numbers. Then, we looked at the TKNN invariant form of topologically protected conductance and saw how it represents a Torus in momentum space. But having the second perturbation of spinning the disk in addition to the magnetic field really complicated the matter and we had to tackle it with a different approach. So we used Many Body Theory and the Diagrammatic Perturbation Theory and finally got a nice expression for the conductance in a spinning Quantum Hall disk.

The work covered in this thesis is for the continuum case and now we are doing it for the discrete case too by applying a similar exercise for a 2D lattice. To verify the conductance expression for the discrete model we plan to simulate this spinning Quantum Hall system on a discrete lattice in future.

2 Introduction to Quantum Hall Effect

(Formulation in Girvin and Goerbig notes)

3 Quantum Hall Effect in disk geometry

So to solve Quantum hall effect in a spinning disk, it would be a good practice to first solve the simple Quantum hall effect without spinning in polar coordinates. We would solve it using two approaches, by series solution and by the operator approach. And after that we would solve QHE in a spinning with both of these approaches too. Note that we won't be solving with a confining potential at the edges and assume a kind of infinite disk, due to this assumption we would be able to extract nice looking exact solutions.

3.1 Series Solution

We have our Hall effect Hamiltonian where we are using the symmetric gauge, $A = \frac{1}{2}Br$.

$$H = \frac{p_r^2}{2m^*} + \frac{1}{2m^* r^2}\left(p_\theta - \frac{eAr}{c}\right)^2 \tag{3.1}$$

We would make the substitution of $w_c = \frac{eB}{mc}$, and apply this Hamiltonian on our wavefunction, $\psi = R_{em}e^{im\theta}$.

$$H\psi = E\psi \tag{3.2}$$

So we get the equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} - \frac{m^{*2}r^2w_c^2}{4\hbar^2} + \frac{mm^*w_c}{\hbar} + \frac{2m^*}{\hbar^2}E\right)R_{em} = 0 \tag{3.3}$$

We will do a simple change of variables with $r = \left(\frac{\hbar}{m^* w_c}\right)^{1/2} y$. After substitution the above equation in terms of y becomes:

$$\left(\frac{\partial^2}{\partial y^2} + \frac{1}{y}\frac{\partial}{\partial y} - \frac{m^2}{y^2} - \frac{y^2}{4} + (m + 2\epsilon)\right)R_{em} = 0 \tag{3.4}$$

Where $\epsilon = \frac{E}{\hbar w_c}$.

Now we replace R_{em} with the Laughlin's trial wave function $R_{em} = y^m e^{-fracy^2 4}U(y)$. After simplifying we are left with:

$$U''(y) + U'(y)\left[\frac{(2m+1)}{y} - y\right] + U(y)\left[(2\epsilon - 1)\right] = 0 \tag{3.5}$$

Now I will take the series solution.

$$U(y) = \sum_{r=0}^{\infty} C_r y^r \tag{3.6}$$

$$U'(y) = \sum_{r=0}^{\infty} r C_r y^{r-1} \tag{3.7}$$

$$U''(y) = \sum_{r=0}^{\infty} r(r-1) C_r y^{r-2} \tag{3.8}$$

We would plug them in and get the following recursion relation:

$$C_{r+2} = C_r \frac{(r+1-2\epsilon)}{(r+2)(r+2+2m)} \tag{3.9}$$

Note that r can only be even and by taking r=2n we can recover the energy expression from this recursion relation:

$$\epsilon = (n + 1/2) \tag{3.10}$$

and by making the substitution, $\epsilon = \frac{E}{\hbar w_c}$, we recover the energy relationship of Quantum hall effect that we had before:

$$E = (n + 1/2)\hbar w_c \tag{3.11}$$

Through this recursion relation we can get the complete set of wavefunctions too that describe the system.

3.2 Operator Approach

Now we would solve the Quantum Hall effect using the operator approach but we would do it by going from polar coordinates. Why do we take this arduous route and overly complicate our lives? Just because we could solve for the spinning disk problem with this approach.

So we would go from polar coordinates to the complex coordinates and once we have done that we would go from the complex coordinates to operators.

Complex coordinates:

$$z = re^{-i\theta} \tag{3.12}$$

$$z^* = re^{i\theta} \tag{3.13}$$

$$r^2 = zz^* \tag{3.14}$$

$$e^{i\theta} = \left(\frac{z^*}{z}\right) \tag{3.15}$$

$$e^{-i\theta} = \left(\frac{z}{z^*}\right) \tag{3.16}$$

Our polar coordinates in terms of complex coordinates can be found by applying chain rule. I would just state the final transformation:

$$\frac{\partial f}{\partial r} = \left(\frac{z}{z^*}\right)\frac{\partial f}{\partial z} + \left(\frac{z^*}{z}\right)\frac{\partial f}{\partial z^*} \tag{3.17}$$

$$\frac{\partial^2 f}{\partial r^2} = 2\frac{\partial^2 f}{\partial z \partial z^*} + \frac{z}{z^*}\frac{\partial^2 f}{\partial z^2} + \frac{z^*}{z}\frac{\partial^2 f}{\partial z^{*2}} \tag{3.18}$$

$$\frac{\partial f}{\partial \theta} = -i\left(z\frac{\partial f}{\partial z} - z^*\frac{\partial f}{\partial z^*}\right) \tag{3.19}$$

$$\frac{\partial^2 f}{\partial \theta^2} = 2zz^*\frac{\partial^2 f}{\partial z \partial z^*} - z^*\frac{\partial f}{\partial z^*} - z\frac{\partial f}{\partial z} - z^{*2}\frac{\partial^2 f}{\partial z^{*2}} - z^2\frac{\partial^2 f}{\partial z^2} \tag{3.20}$$

So I will take my Hamiltonian where we are using the symmetric gauge, $A = \frac{1}{2}Br$.

$$H = \frac{p_r^2}{2m^*} + \frac{1}{2m^* r^2}\left(p_\theta + \frac{eAr}{c}\right)^2 \tag{3.21}$$

We will make the following substitutions; $w_c = \frac{eB}{mc}$ and $l^2 = \frac{\hbar c}{eB}$, where w_c is the cyclotron frequency and l is the magnetic length. We then make the transformation to make my coordinates dimensionless, $r = lr'$, after simplifying and relabeling I get the following:

$$H = \frac{\hbar w}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - i\frac{\partial}{\partial \theta} + \frac{r^2}{4}\right) \tag{3.22}$$

Now plug in the transformations, and we get our Hamiltonian in complex coordinates:

$$\frac{\hbar w}{2}\left(-4\frac{\partial^2}{\partial z \partial z^*} - z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*} + \frac{zz^*}{4}\right) \tag{3.23}$$

Now I will define my operators in terms of polar coordinates:

$$z = re^{-i\theta} = x - iy \tag{3.24}$$

$$z^* = re^{i\theta} = x + iy \tag{3.25}$$

$$b = \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} + 2\frac{\partial}{\partial z}\right) \tag{3.26}$$

$$b^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{2} - 2\frac{\partial}{\partial z^*}\right) \tag{3.27}$$

$$a^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z^*}{2} - 2\frac{\partial}{\partial z}\right) \tag{3.28}$$

$$a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right) \tag{3.29}$$

Where they satisfy the following commutation relationships:

$$[a, a^\dagger] = 1 \tag{3.30}$$

$$[b, b^\dagger] = 1 \tag{3.31}$$

$$[a, b] = 0 \tag{3.32}$$

$$[a^\dagger, b^\dagger] = 0 \tag{3.33}$$

$$aa^\dagger = 1 + a^\dagger a \tag{3.34}$$

$$bb^\dagger = 1 + b^\dagger b \tag{3.35}$$

Now replace the complex coordinates with these operators in the Hamiltonian and you get

$$H = \hbar w_c[a^\dagger a + 1/2] \tag{3.36}$$

So when we apply this hamiltonian to our eigenstate $|n, m\rangle$:

$$H|n, m\rangle = E_n|n, m\rangle \tag{3.37}$$

And I would get the energy relationship for the simple quantum hall effect

$$E_n = (n + 1/2)\hbar w_c \tag{3.38}$$

4 Spinning Disk without magnetic field

So before we move on to the case of a spinning disk with magnetic field, we would first discuss about a simple spinning disk without the magnetic field. First we set up our coordinates like you can see in figure.

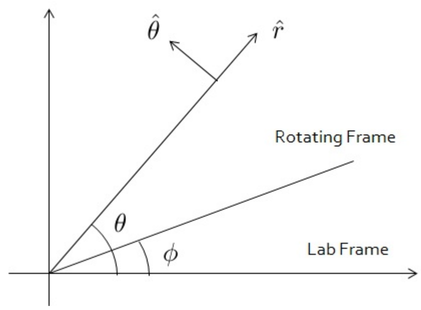

Figure 1: setting up our coordinates for a rotating disk

$$\vec{r} = r\hat{r} \tag{4.1}$$

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \tag{4.2}$$

$$= \dot{r}\hat{r} + r(\dot{\theta} + \dot{\phi})\hat{\theta} \tag{4.3}$$

where $\frac{d\vec{r}}{dt} = (\dot{\theta} + \dot{\phi})\hat{\theta}$. First we write the Lagrangian for this system and ignore the potential term.

$$L = \frac{1}{2}m\vec{v}^2 - V \tag{4.4}$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 + 2r^2\dot{\theta}\dot{\phi}) \tag{4.5}$$

Now that we have our lagrangian we can find the corresponding hamiltonian too.

$$H = P_r\dot{r} + P_\theta\dot{\theta} - L \tag{4.6}$$

$$= \frac{P_r^2}{2m} + \frac{P_\theta^2}{2m} - \Omega P_\theta \tag{4.7}$$

Here $\Omega = \dot{\phi}$ and represents the frequency of rotation. We can expand this hamiltonian in terms of its operators. Note that we have labeled the mass as m^* to differentiate it with the m in the wave function.

$$H = -\frac{\hbar^2}{2m^*}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right] + i\hbar\frac{\partial}{\partial \theta}\Omega \tag{4.8}$$

We then take a trial wavefunction $\psi = e^{-im\theta}R(r)$ and apply our Hamiltonian on it, $H\psi = E\psi$.

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{2m^*E}{\hbar^2} - \frac{2m^*m\Omega}{\hbar} - \frac{m^2}{r^2}\right]R(r) = 0 \tag{4.9}$$

Now we can compare and see that the above equation has the form of a Bessel equation:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + (K^2 - \frac{m^2}{r^2})\right]R(r) = 0 \tag{4.10}$$

So our wavefunction has solutions of the form of the Bessel equations:

$$R(r) = A\ J_m(kr) + B\ Y_m(kr) \tag{4.11}$$

However we take $B = 0$, otherwise there would be a singularity at $r = 0$.

Now we would find the energies for our spinning disk, $K^2 = \frac{2m^*E}{\hbar^2} - \frac{2m^*m\Omega}{\hbar}$ and $Kr_2 = \alpha_{m,n}$.

$\alpha_{m,n}$ represents the n^{th} root of the m^{th} order Bessel function. Now by making E as the subject of the formula we can find the energies of the spinning disk:

$$E = \frac{\hbar^2\alpha_{m,n}^2}{2mr_2^2} + m\hbar\Omega \tag{4.12}$$

5 Quantum Hall Effect in Spinning Disk Geometry

Now finally we would consider a spinning disk with a magnetic field. We would again use both the operator approach and the series solution approach to get the same result, we would use the transformations we defined in the sections above.

5.1 Series Solution

Again we consider the rotating frame as we did in the earlier section, but this time our lagrangian would be different because of the magnetic field.

First we set up our coordinates like you can see in figure.

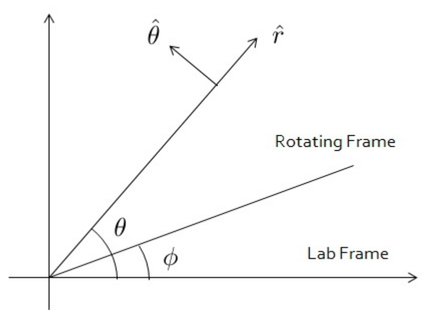

Figure 2: setting up our coordinates for a rotating disk

$$\vec{r} = r\hat{r} \tag{5.1}$$

$$\frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} \tag{5.2}$$

$$= \dot{r}\hat{r} + r(\dot{\theta} + \dot{\phi})\hat{\theta} \tag{5.3}$$

where $\frac{d\vec{r}}{dt} = (\dot{\theta} + \dot{\phi})\hat{\theta}$. First we write the Lagrangian for this system and ignore the potential term.

$$L = \frac{1}{2}m\vec{v}^2 + \frac{e}{c}\vec{v}.\vec{A} - V \tag{5.4}$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 + 2r^2\dot{\theta}\dot{\phi}) + \frac{er}{c}(\dot{\theta} + \dot{\phi})\vec{A} - V \tag{5.5}$$

Now that we have our lagrangian we can find the corresponding hamiltonian too.

$$H = P_r\dot{r} + P_\theta\dot{\theta} - L \tag{5.6}$$
$$= \frac{P_r^2}{2m^*} + \frac{P_\theta^2}{2m^*} - \Omega P_\theta - \frac{P_\theta w_c}{2} + \frac{m^* r^2 w_c^2}{8} \tag{5.7}$$

Note that this is same as before for the non-spinning except for the ΩP_θ term where Ω is the rotation frequency and this Hamiltonian will reduce to the Hamiltonian of the non-spinning disk with magnetic field if we take Ω to be zero.

We would make the substitution of $w_c = \frac{eB}{mc}$, and apply this Hamiltonian on our wavefunction, $\psi = R_{em}e^{im\theta}$.

$$H\psi = E\psi \tag{5.8}$$

We will do a simple change of variables with $r = \left(\frac{\hbar c}{m^* w_c}\right)^{1/2} y$. After substitution the above equation in terms of y becomes:

$$\left(\frac{\partial^2}{\partial y^2} + \frac{1}{y}\frac{\partial}{\partial y} - \frac{m^2}{y^2} - \frac{y^2}{4} + (m + \frac{2m\Omega}{w_c} + 2\epsilon)\right) R_{em} = 0 \tag{5.9}$$

Where $\epsilon = \frac{E}{\hbar w_c}$.

Note again that we only have a difference of $\frac{2m\Omega}{w_c}$ factor which is also rotational frequency dependent, compared to the non-spinning disk case.

Now we replace R_{em} with the Laughlin's trial wave function $R_{em} = y^m e^{-fracy^2 4} U(y)$. After simplifying we are left with:

$$U''(y) + U'(y)\left[\frac{(2m+1)}{y} - y\right] + U(y)\left[(2\epsilon - 1 - \frac{2m\Omega}{w_c})\right] = 0 \tag{5.10}$$

Now I will take the series solution.

$$U(y) = \sum_{r=0}^{\infty} C_r y^r \tag{5.11}$$
$$U'(y) = \sum_{r=0}^{\infty} rC_r y^{r-1} \tag{5.12}$$
$$U''(y) = \sum_{r=0}^{\infty} r(r-1)C_r y^{r-2} \tag{5.13}$$

We would plug them in and get the following recursion relation:

$$C_{r+2} = C_r \frac{(r + 1 + \frac{2m\Omega}{w_c} - 2\epsilon)}{(r + 2)(r + 2 + 2m)} \tag{5.14}$$

Note that r can only be even and by taking r=2n we can recover the energy expression from this recursion relation:

$$\epsilon = (n + 1/2) + \frac{m\Omega}{w_c} \tag{5.15}$$

and by making the substitution, $\epsilon = \frac{E}{\hbar w_c}$, we recover the energy relationship of Quantum hall effect that we had before:

$$E = (n + 1/2)\hbar w_c + \hbar m\Omega \tag{5.16}$$

5.2 Operator Approach

Now we will find the quantum hall energies in the rotating coordinates. we would again repeat the whole exercise of coordinate transformation and all. I would write the transformations again, you can refer to them from the section of operator approach solution to spinning disk with magnetic field.

So we have our Hamiltonian for a rotating disk in a magnetic field that we calculate in the previous subsection.

$$H = P_r \dot{r} + P_\theta \dot{\theta} - L \tag{5.17}$$

$$= \frac{P_r^2}{2m^*} + \frac{P_\theta^2}{2m^*} - \Omega P_\theta - \frac{P_\theta w_c}{2} + \frac{m^* r^2 w_c^2}{8} \tag{5.18}$$

We will make the following substitutions; $w_c = \frac{eB}{mc}$ and $l^2 = \frac{\hbar c}{eB}$, where w_c is the cyclotron frequency and l is the magnetic length. We then make the transformation to make my coordinates dimensionless, $r = lr'$, after simplifying and relabeling I get the following:

$$H = \frac{\hbar w}{2}\left(-\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} - 2i\frac{\Omega}{w_c}\frac{\partial}{\partial \theta} - i\frac{\partial}{\partial \theta} + \frac{r^2}{4}\right) \tag{5.19}$$

Now plug in the transformations, and we get our Hamiltonian in complex coordinates:

$$\frac{\hbar w}{2}\left(-4\frac{\partial^2}{\partial z \partial z^*} - z\frac{\partial}{\partial z} + z^*\frac{\partial}{\partial z^*} + \frac{zz^*}{4} + 2\frac{\Omega}{w_c}\left[z^*\frac{\partial}{\partial z^*} - z\frac{\partial}{\partial z}\right]\right) \tag{5.20}$$

Now replace the complex coordinates with the operators, defined in the previous section, in the Hamiltonian and you get:

$$H = \hbar w_c \left[a^\dagger a + 1/2 - \frac{\Omega}{w_c}[b^\dagger b - a^\dagger a]\right] \tag{5.21}$$

Here $[b^\dagger b - a^\dagger a]$ is just the $\hat{L_z}$ or the Angular momentum operator.

So when we apply this hamiltonian to our eigenstate $|n, m\rangle$:

$$H|n, m\rangle = E_n|n, m\rangle \tag{5.22}$$

I would get the energy relationship for the quantum hall effect in a spinning disk

$$E_n = (n + 1/2)\hbar w_c + \Omega \hbar m \tag{5.23}$$

Which agrees with the result of we got from the series solution approach.

6 Quantum Hall Effect in spinning disk and the Percolation model

Now that we have this new energy relation what does this mean. So without spinning you had degenerate levels, for each Landau level n, there would be m degenerate levels that all had the same energy. But when there are impurities present in the sample, because of the potential due to impurities the degeneracy of the Landau level is lifted.

So the impurities cause a broadening of peaks at the Landau level. As you increase the number of impurities your peaks would become more broadened. Impurities also play an important role in the plateau formation of quantized conductance, however the exact mechanism is still not understood.

By spinning the disk, we also get a broadening of peaks as the degeneracy in m is lifted. Interestingly this broadening can be controlled by the frequency of spinning. So what we propose is that by spinning the disk you can mimic the broadening caused by impurities, and now you can also vary your broadening by changing your spinning frequency and see how the conductance plateaus change. In this way we might be better able to understand the relation between conductance and the impurities in the sample.

We have the energy relations because of spinning, now we would work on getting the conductance relation for a spinning disk with magnetic field.

7 Kubo's Formula

We use perturbation theory, we have taken $\hbar$ to be 1

$$H = H_o + H'(t) \tag{7.1}$$

Recall, where H is time independent

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle \tag{7.2}$$

$$H_o|l\rangle = \varepsilon_l|l\rangle \tag{7.3}$$

Let'a define

$$|\psi(t)\rangle_I = e^{-iH_o t}|\psi(t)\rangle \tag{7.4}$$

Take derivatives on both side and multiply it by i

$$\begin{aligned} i\frac{d}{dt}|\psi(t)\rangle_I &= i\frac{d}{dt}e^{iH_o t}|\psi(t)\rangle + e^{iH_o t}i\frac{d}{dt}|\psi(t)\rangle &(7.5)\\ &= -H_o e^{iH_o t}|\psi(t)\rangle + e^{iH_o t}(H_o + H')|\psi(t)\rangle &(7.6)\\ &= e^{iH_o t}H'(t)|\psi(t)\rangle &(7.7)\end{aligned}$$

$$e^{iH_o t}H'(t)e^{-iH_o t}e^{iH_o t}|\psi(t)\rangle = e^{iH_o t}H'(t)e^{-iH_o t}|\psi(t)\rangle_I \tag{7.8}$$

Let's define $e^{iH_o t}H'(t)e^{-iH_o t}$ as $H_I(t)$.

Now we solve the first order Ordinary Differential Equation:

$$i\frac{d}{dt}|\psi(t)\rangle_I = H_I(t)|\psi(t)\rangle_I \tag{7.9}$$

We need Boundary Conditions.
We have turned on the perturbation at time t=-T

$$|\psi(t)\rangle_I = |l\rangle \tag{7.10}$$

Solution:

$$|\psi(t)\rangle_I = e^{-i\int_{-T}^{t} dt' H_I(t')}|l\rangle \tag{7.11}$$

As $H_I(t')$ is an operator. We expand the exponential and get $1-i\int_{-T}^{t} dt' H_I(t')+$ ..... H_I at different times donot commute with each other. But we don't need the second order and higher terms for now so we can ignore them.

$$|\psi(t)\rangle_I = \left\{1 - i\int_{-T}^{t} dt' H_I(t')\right\}|l\rangle + higher\ order\ terms \tag{7.12}$$

Recall:

$$\begin{aligned}
|\psi(t)\rangle &= e^{-iH_o t}|\psi(t)\rangle_I &(7.13)\\
&= e^{-iH_o t}\left\{1 - i\int_{-T}^{t} dt' H_I(t')\right\}|l\rangle &(7.14)\\
&= e^{-iH_o t}\left\{1 - i\int_{-T}^{t} dt' e^{iH_o t'} H'(t') e^{-iH_o t'}\right\}|l\rangle &(7.15)\\
&= e^{-iH_o t}\left\{1 - i\int_{-T}^{t} dt' e^{iH_o t'} \sum_{l'} |l'\rangle\langle l'|\ H'(t') e^{-iH_o t'}\right\}|l\rangle &(7.16)\\
&= e^{-i\varepsilon_l t}|l\rangle - i\sum_{l'}\int_{-T}^{t} dt' e^{-iH_o t}|l'\rangle e^{i\varepsilon_{l'} t'}\langle l'|H'(t')|l\rangle e^{-i\varepsilon_{l'} t} &(7.17)
\end{aligned}$$

Here $\langle l'|H'|l\rangle$ is the matrix element of perturbation.

Let $H'(t') = H'e^{-iwt'} + H'^{\dagger}e^{iwt'}$ where the key property is that H' is Hermitian. And 'w' is the frequency of the radiation coming in.

So we first plug in only the first part of our perturbation Hamiltonian and get:

$$e^{-i\varepsilon_l t}|l\rangle - i\sum_{l'}|l'\rangle\int_{-T}^{t} dt' e^{i(\varepsilon_{l'} - w - \varepsilon_l)t'} e^{i\varepsilon_{l'} t'}\langle l'|H'(t')|l\rangle e^{-i\varepsilon_{l'} t} \tag{7.18}$$

Now as $\langle l'|H'(t')|l\rangle$ is time independent, and as $-T \to \infty$. The integral becomes:

$$\int_{-T}^{t} dt' e^{i(\varepsilon_{l'} - w - \varepsilon_l)t'} = \frac{e^{i(\varepsilon_{l'} - w - \varepsilon_l)t'}}{i(\varepsilon_{l'} - w - \varepsilon_l)}|_{-\infty}^{t} \tag{7.19}$$

As you can see this integral is undefined at $t' = -\infty$ so to define it we add a small imaginary part to w.

$$w \to w + i\eta\ where\ \eta \to 0^+ \tag{7.20}$$

Now the exponential dies out at $t' = -\infty$ and our expression becomes:

$$= e^{-i\varepsilon_l t}|l\rangle - \sum_{l'} |l'\rangle \frac{e^{i(\varepsilon_{l'} - \varepsilon_l - w)t}}{\varepsilon_{l'} - \varepsilon_l - w} \langle l'|H'|l\rangle e^{-i\varepsilon_{l'} t} \qquad = -\sum_{l'} |l'\rangle \frac{1}{\varepsilon_{l'} - \varepsilon_l - w} \tag{7.21}$$

Also consider the other part of H' where $H'^{\dagger} = H'$ and we finally get:

$$|\psi(t)\rangle = e^{-i\varepsilon_l t} \left\{ |l\rangle - \sum_{l'} |l'\rangle \frac{\langle l'|H'|l\rangle}{\varepsilon_{l'} - \varepsilon_l - w} e^{-iwt} - \sum_{l'} |l'\rangle \frac{\langle l'|H'|l\rangle}{\varepsilon_{l'} - \varepsilon_l + w^*} e^{iw^* t} \right\} \tag{7.22}$$

$$= e^{-i\varepsilon_l t} \{ |l\rangle + |\delta\rangle \} \tag{7.23}$$

Now we consider a more specific case in which we have a perpendicular magnetic field passing through our 2DEG system. We apply some radiation on it now.

$$E = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t} \tag{7.24}$$

So our Hamiltonian becomes

$$\frac{1}{2m}\left(p - \frac{eA}{c}\right)^2 + e\varphi \tag{7.25}$$

we put an external Electric field

$$\vec{E} = \vec{E_o} e^{-iwt} \tag{7.26}$$

For a constant E field

$$\vec{E} = \frac{-1}{c}\frac{\partial \vec{A}}{\partial t} \vec{A} = -\vec{E_o}\frac{ic}{w} e^{-iwt} \tag{7.27}$$

And so our Hamiltonian becomes:

$$H = \frac{p^2}{2m} - \frac{a\vec{A}.\vec{p}}{mc} + O\left(A^2\right) \tag{7.28}$$

$$= \frac{p^2}{2m} + \frac{ie}{mw}\vec{E_o}.\vec{p} e^{-iwt} + O\left(A^2\right) \tag{7.29}$$

$$= H_o + H'(t) \tag{7.30}$$

Here we consider $\frac{p^2}{2m}$ to be H_o, and $H'(t) = \frac{ie}{mw}\vec{E_o}.\vec{p}e^{-iwt}$, and we ignore the higher order terms.

We plug it in our Kubo's formula and get:

$$|\psi(t)\rangle = e^{-i\varepsilon_l t}\left\{|l\rangle - \frac{ie}{mw}E_j\sum_{l'}|l'\rangle\frac{\langle l'|p_j|l\rangle}{\varepsilon_{l'}-\varepsilon_l - w}e^{-iwt} + \frac{ie}{mw^*}E_j\sum_{l'}|l'\rangle\frac{\langle l'|p_j|l\rangle}{\varepsilon_{l'}-\varepsilon_l + w^*}e^{iw^*t}\right\} \tag{7.31}$$

Electron started in state $|l\rangle$ at $t = -\infty$ and now it is this. What we need is the current in this state at time t. We need to find the current density operator j_i.

$$j_i = ev_i = e\dot{x}_i = e\frac{\partial H}{\partial p_i}e\frac{\partial H}{\partial p_i} = \frac{e}{m}p_i - \frac{eA_i}{c} \tag{7.32}$$

Plug in the value of A_i. In this expression i = x,y directions.

$$\frac{e}{m}p_i + \frac{ie^2}{mw}E_i e^{-iwt} \tag{7.33}$$

We take the expectation value of the current operator

$$\langle\psi(t)|j_i|\psi(t)\rangle = \langle\psi(t)|\frac{e}{m}p_i|\psi(t)\rangle + \frac{ie^2}{mw}E_i e^{-iwt}\langle\psi(t)|\psi(t)\rangle \tag{7.34}$$

We can define the current density in terms of the conductivity tensor

$$j_i = \sigma_{ij}E_j \tag{7.35}$$

$$j_i e^{-iwt} = \sigma_{ij}E_j e^{-iwt} \tag{7.36}$$

Only considering the e^{-iwt} for now. Plug equation dash in dash and we get.

$$\langle\psi|j_i|\psi\rangle = \frac{ie^2}{mw}E_i e^{-iwt} + \langle l|\frac{ep_i}{m}|\delta\rangle + \langle\delta|\frac{ep_i}{m}|l\rangle + \langle l|\frac{ep_i}{m}|l\rangle + H.O\ terms \tag{7.37}$$

$$\langle\psi|j_i|\psi\rangle = \frac{ie^2}{mw}E_j e^{-iwt}\left[\delta_{ij} - \frac{1}{m}\sum_{l'}\left\{\frac{\langle l|\ p_i\ |l'\rangle\langle l'|\ p_j\ |l\rangle}{\varepsilon_{l'}-\varepsilon_l - w} + \frac{\langle l|\ p_j\ |l'\rangle\langle l'|\ p_i\ |l\rangle}{\varepsilon_{l'}-\varepsilon_l + w}\right\}\right] \tag{7.38}$$

l' represent the unoccupied states and l represent occupied states.

$$\sigma_{ij} = \frac{ie^2}{mw}\left[\delta_{ij} - \frac{1}{m}\sum_{l'}\left\{\frac{\langle l|\ p_i\ |l'\rangle\langle l'|\ p_j\ |l\rangle}{\varepsilon_{l'} - \varepsilon_l - w} + \frac{\langle l|\ p_j\ |l'\rangle\langle l'|\ p_i\ |l\rangle}{\varepsilon_{l'} - \varepsilon_l + w}\right\}\right] \tag{7.39}$$

Let l_F be the state corresponding to the Fermi energy level. We define a function f_l

$$f_l = 1\ if\ l \leq l_f f_l = 0\ if\ l > l_f \tag{7.40}$$

We sum over all electrons in the occupied levels.

$$\sigma_{ij} = \frac{ie^2}{mw}\left[\sum_l f_l\delta_{ij} - \frac{1}{m}\sum_{l'\neq l} f_l\left\{\frac{\langle l|\ p_i\ |l'\rangle\langle l'|\ p_j\ |l\rangle}{\varepsilon_{l'} - \varepsilon_l - w} + \frac{\langle l|\ p_j\ |l'\rangle\langle l'|\ p_i\ |l\rangle}{\varepsilon_{l'} - \varepsilon_l + w}\right\}\right] \tag{7.41}$$

For the second term relabel $l = l'$ and $l' = l$.

$$\sigma_{ij} = -\frac{ie^2}{m^2w}\left[\sum_{ll'}(f_l - f_{l'})\frac{1}{m}\sum_{l'\neq l}\frac{\langle l|\ p_i\ |l'\rangle\langle l'|\ p_j\ |l\rangle}{\varepsilon_{l'} - \varepsilon_l - w}\right] \tag{7.42}$$

Now there's a problem, for static conductivity, w = o and the whole thing would blow up. So we do an expansion:

$$\frac{1}{\varepsilon_{l'} - \varepsilon_l - w} = \frac{1}{(\varepsilon_{l'} - \varepsilon_l)\left[1 - \frac{w}{\varepsilon_{l'} - 1\varepsilon_l}\right]} \tag{7.43}$$

$$= \frac{1}{(\varepsilon_{l'} - \varepsilon_l)} + \frac{w^2}{(\varepsilon_{l'} - \varepsilon_l)^2} + \cdots \tag{7.44}$$

After plugging in, the first term becomes:

$$\sigma_{ij} = -\frac{ie^2}{w}\left[\sum_{ll'}(f_l - f_{l'})\frac{\langle l|\ v_i\ |l'\rangle\langle l'|\ v_j\ |l\rangle}{\varepsilon_{l'} - \varepsilon_l}\right] \tag{7.45}$$

This term blows up but we would use a nice trick. ($\hbar = 1$)

$$\frac{1}{i}\frac{d}{dt}\hat{O} = \left[H_o, \hat{O}\right] v_x = \frac{dx}{dt} = i\left[H_o, x\right] \tag{7.46}$$

$$\langle l|\ v_x\ |l'\rangle = \langle l|\ H_o x - xH_o\ |l'\rangle \tag{7.47}$$

$$= i(\varepsilon_l - \varepsilon_{l'})\langle l|\ x\ |l'\rangle \tag{7.48}$$

Plugging in

$$\sigma_{xy} = \frac{e^2}{w} \sum f_l \langle l|\ x\ |l'\rangle\langle l'|\ v_y\ |l\rangle - \frac{e^2}{w} \sum f_{l'} \langle l|\ x\ |l'\rangle\langle l'|\ v_y\ |l\rangle \tag{7.49}$$

$$= \frac{e^2}{w} \sum f_l \langle l|\ xv_y\ |l\rangle - \frac{e^2}{w} \sum f_{l'} \langle l'|\ v_y x\ |l'\rangle \tag{7.50}$$

$$= \frac{e^2}{w} \sum f_l \langle l|\ xv_y - v_y x\ |l\rangle \tag{7.51}$$

$$= 0 \tag{7.52}$$

The first term goes to zero as the commutator $[x, v_y] = 0$.

The second term is:

$$\sigma_{xy} = -\frac{ie^2}{A_o \hbar} \sum_{ll'} (f_l - f_{l'}) \frac{\langle l|\ v_x\ |l'\rangle\langle l'|\ v_y\ |l\rangle}{(\varepsilon_{l'} - \varepsilon_l)^2} \tag{7.53}$$

Where A_o is the area of the sample.

8 Kubo's Formula for spinning disk

Now we would try to find a similar formula like Kubo formula for a spinning disk but with no magnetic field. So the Hamiltonian with rotation is:

$$H = \frac{p^2}{2m} + U(x, y) - \vec{\Omega} L \tag{8.1}$$

Where $\vec{\Omega} = \hat{z}\Omega e^{\eta t}$ and $L = r \times p$. and our wavefunction is $\psi = e^{ikx}u_k$.

So

$$H\psi = e^{ikx}\left[\frac{(p + \hbar k)^2}{2m} + U(x, y) - \Omega r \times (p + \hbar k)\right] u_k \tag{8.2}$$

Here $\Omega\ r \times (p + \hbar k)$ is the perturbation $H'(t)$. So we would use equation () from the previous section and plug the perturbation in.

$$|\psi(t)\rangle = e^{-i\varepsilon_\alpha t}\left(|\alpha\rangle - \sum_\beta |\beta\rangle \frac{\langle\beta|H'|\alpha\rangle}{\varepsilon_\beta - \varepsilon_\alpha - i\eta}\right) \tag{8.3}$$

And there is also the similar hermitian term added to the expression above but we would ignore it for now. We can define $L_3 = [r \times (p + \hbar k)]$. So our wavefunction becomes:

$$|\psi(t)\rangle = e^{-i\varepsilon_\alpha t}\left(|\alpha\rangle - \Omega\ \sum_\beta |\beta\rangle \frac{\langle\beta|L_3|\alpha\rangle}{\varepsilon_\beta - \varepsilon_\alpha - i\eta}\right) \tag{8.4}$$

$$\langle\psi|v_i|\psi\rangle = \langle\alpha|v_i|\alpha\rangle - \Omega \sum \frac{\langle\alpha|v_i|\beta\rangle\langle\beta|L_3|\alpha\rangle}{\varepsilon_\beta - \varepsilon_\alpha - i\eta} - \Omega \sum \frac{\langle\alpha|L_3|\beta\rangle\langle\beta|v_i|\alpha\rangle}{\varepsilon_\beta - \varepsilon_\alpha + i\eta} + \Omega^2 \cdots \tag{8.5}$$

As $\langle\alpha|v_i|\alpha\rangle = 0$ and we also ignore the Ω^2 and higher order terms, we are left with:

$$\langle\psi|v_i|\psi\rangle = -\Omega \sum \left\{\frac{\langle\alpha|v_i|\beta\rangle\langle\beta|L_3|\alpha\rangle}{\varepsilon_\beta - \varepsilon_\alpha - i\eta} - \frac{\langle\alpha|L_3|\beta\rangle\langle\beta|v_i|\alpha\rangle}{\varepsilon_\beta - \varepsilon_\alpha + i\eta}\right\} \tag{8.6}$$

We will use the following relations which we derived in the previous section

$$\langle\alpha|v_i|\beta\rangle = -(\varepsilon_\beta - \varepsilon_\alpha)\frac{1}{\hbar}\langle\frac{\partial\alpha}{\partial k_i}|\beta\rangle \tag{8.7}$$

$$\langle\beta|v_i|\alpha\rangle = -(\varepsilon_\beta - \varepsilon_\alpha)\frac{1}{\hbar}\langle\beta|\frac{\partial\alpha}{\partial k_i}\rangle \tag{8.8}$$

After plugging them in we get:

$$\langle\psi|v_i|\psi\rangle = -\Omega\sum\left\{-\frac{1}{\hbar}\langle\frac{\partial\alpha}{\partial k_i}|\beta\rangle\langle\beta|L_3|\alpha\rangle - \frac{1}{\hbar}\langle\alpha|L_3|\beta\rangle\langle\beta|\frac{\partial\alpha}{\partial k_i}\rangle\right\} \tag{8.9}$$

Now we use $\sum_\beta\ |\beta\rangle\langle\beta| = 1 - \sum_{\alpha'}|\alpha'\rangle\langle\alpha'|$

$$\langle\psi|v_i|\psi\rangle = \frac{\Omega}{\hbar}\left\{\langle\frac{\partial\alpha}{\partial k_i}|\ (\ 1 - \sum_{\alpha'}|\alpha'\rangle\langle\alpha'|\ )\ |L_3|\alpha\rangle + \langle\alpha|L_3|\ (\ 1 - \sum_{\alpha'}|\alpha'\rangle\langle\alpha'|\ )\ |\frac{\partial\alpha}{\partial k_i}\rangle\right\} \tag{8.10}$$

$$\langle\psi|v_i|\psi\rangle = \frac{\Omega}{\hbar}\left\{\langle\frac{\partial\alpha}{\partial k_i}|L_3|\alpha\rangle + \langle\alpha|L_3|\frac{\partial\alpha}{\partial k_i}\rangle\right\} - \frac{\Omega}{\hbar}\sum_{\alpha'}\left\{\langle\frac{\partial\alpha}{\partial k_i}|\alpha'\rangle\langle\alpha'|L_3|\alpha\rangle + \langle\alpha|L_3|\alpha'\rangle\langle\alpha'|\frac{\partial\alpha}{\partial k_i}\rangle|\right\} \tag{8.11}$$

The first two terms go to zero. We sum the next two terms on α.

$$\langle\psi|v_i|\psi\rangle = -\frac{\Omega}{\hbar}\sum_{\alpha\alpha'}\left\{\langle\frac{\partial\alpha}{\partial k_i}|\alpha'\rangle\langle\alpha'|L_3|\alpha\rangle + \langle\alpha|L_3|\alpha'\rangle\langle\alpha'|\frac{\partial\alpha}{\partial k_i}\rangle|\right\} \tag{8.12}$$

After relabeling the second term, both of them cancel each other and this also goes to zero.

$$\langle\psi|v_i|\psi\rangle = 0 \tag{8.13}$$

So we have shown that the conductance due to rotation when treated as a perturbation is zero to the first order. This is unexpected but remember that this was done only to the first order and maybe we have non-zero conductance for higher orders. Because of the complexity we won't solve it for higher orders.

9 QHE on a magnetic Bravais Lattice

The schrodinger equation for a 2D non interacting electron system in a uniform magnetic field perpendicular to the plane is written as

$$H\psi = \left[\frac{1}{2m}\left(p - \frac{eA}{c}\right)^2 + U(x, y)\right]\psi = E\psi \tag{9.1}$$

We consider the case where the potential is periodic in both x and y directions on the Bravais lattice.

$$U(x + a, y) = U(x, y + b) = U(x, y) \tag{9.2}$$

The system is invariant under translation by a in x direction and translation by b in the y direction but the Hamiltonian is not invariant. The reason is that although the magnetic field is constant through out the lattice but the gauge potential A is not constant. Now we introduce some formalism:

Let $\vec{R}$ be the Bravais lattice vector

$$\vec{R} = na\hat{x} + mb\hat{y} \tag{9.3}$$

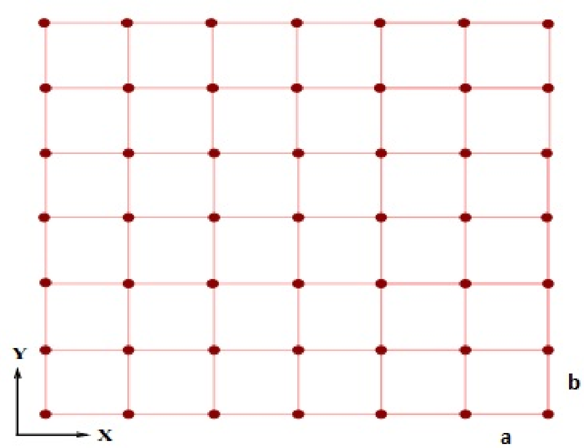

Figure 3: A Bravais lattice with unit cells of size a in the x direction and b in the y direction

For each Bravais lattice vector $\vec{R}$, we define an ordinary translation operator T_R which when acts on a function f(r), it shifts the argument by $\vec{R}$.

$$T_R f(\vec{r}) = f(\vec{r} + \vec{R}) \tag{9.4}$$

The explicit form of this ordinary translation operator is:

$$T_R = e^{\frac{i}{\hbar}\vec{R}\cdot\vec{p}} \tag{9.5}$$

When T_R is applied to a Hamiltonian, the potential U(r) remains invariant. However, the gauge potential $\vec{A}(\vec{r} + \vec{R})$ is generally not equal to $\vec{A}(\vec{r})$.

So we consider the magnetic translation operators

$$\hat{T_R} = exp\frac{i}{\hbar}\vec{R}\cdot[\vec{p} + e(\vec{r}\times\vec{B})/2] \tag{9.6}$$

$$= T_R exp\frac{ie}{\hbar}(\vec{B}\times\vec{R})\cdot\vec{r}/2] \tag{9.7}$$

For the symmetric gauge $\hat{T_R}$ leaves the Hamiltonian invariant. Note that the magnetic translations do not commute with each other for a general case:

$$\hat{T_a}\hat{T_b} = exp(2\pi i\phi)\hat{T_b}\hat{T_a} \tag{9.8}$$

Where $\phi = (eB/h)ab$ is the number of magnetic flux in the unit cell. When ϕ is a rational number, $\phi = p/q$ where p,q are relatively prime integers, we have a subset of translations that do commute.

For this we define an enlarged unit cell which we would call the magnetic unit cell and it would have an integral number of flux quanta passing through it. So our new Bravais lattice would be of the form:

$$\vec{R}' = n(qa)\hat{x} + mb\hat{y} \tag{9.9}$$

So with this formulation p/q flux quanta passed through the ordinary unit cell and now p quanta pass through the magnetic unit cell.

If ψ diagonalizes H and $\hat{T_R}$ simultaneously, then you can show that

$$\hat{T_{qa}}\psi = e^{ik_1 qa}\psi \tag{9.10}$$

$$\hat{T_b}\psi = e^{ik_2 b}\psi \tag{9.11}$$

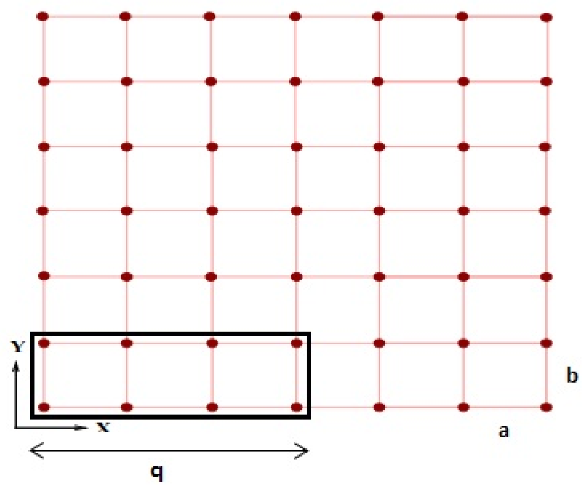

Figure 4: A Bravais lattice with magnetic unit cells of size qa in the x direction and b in the y direction

So we can write our wavefunction in the Bloch form

$$\psi_{k_1k_2}(x,y) = e^{i(k_1x+k_2y)}u_{k_1k_2}(x,y) \tag{9.12}$$

So using this we can define our Bloch conditions:

$$\psi_{k_1k_2}(x+qa,y) = e^{-i\pi py/b}u_{k_1k_2}(x,y) \tag{9.13}$$

$$\psi_{k_1k_2}(x,y+b) = e^{i\pi px/qa}u_{k_1k_2}(x,y) \tag{9.14}$$

We will see if the phase change around the boundary of the magnetic unit cell is gauge invariant. Using the above relations it turns out the phase change is always $2\pi p$.

So our wavefunctions after coming around the boundary is given by:

$$u_{k_1k_2}(x,y) = |u_{k_1k_2}(x,y)|exp[i\theta_{k_1k_2}(x,y)] \tag{9.15}$$

where

$$p = \frac{-1}{2\pi}\int \vec{dl}\cdot\frac{\partial\theta_{k_1k_2(x,y)}}{\vec{dl}} \tag{9.16}$$

$\int \vec{dl}$ represents a counterclockwise line integral around the boundary of the magnetic unit cell. Here p is a topologically invariant number, we would use this result in the next section when we discuss topologically protected conductance.

10 TKNN Invariance in QHE and relation to Topology

We would use the Kubo's formula we derived in the previous section and work on it to relate it to Topology as done by Kohmoto in his paper.

Now we include a periodic potential in our Hamiltonian too.

$$\frac{1}{2m}\left(p - \frac{eA}{c}\right)^2 + U(x,y) \tag{10.1}$$

Our wavefunctions would now be in the form of Bloch wavefunction because of the periodicity.

$$\psi = e^{ikx}u_k(x) \tag{10.2}$$

We can see how the momentum operator $\hat{p}$ acts on the wavefunction:

$$\hat{p}(e^{ikx}u_k(x)) = e^{ikx}(p + \hbar k)u_k(x) \tag{10.3}$$

Now we apply the whole Hamiltonian on our wavefunction. $H\psi = \varepsilon_\alpha \psi$

$$\left[\frac{1}{2m}\left(p - \frac{eA}{c}\right)^2 + U\right] e^{ikx}u_k(x) = \left[\frac{1}{2m}\left(p + \hbar k - \frac{eA}{c}\right)^2 + U\right] e^{ikx}u_k(x) \tag{10.4}$$

Now we need to calculate the velocity, classically it is:

$$v_x = \frac{1}{\hbar}\frac{\partial H}{\partial k_x} \tag{10.5}$$

$$= \frac{(p_x + \hbar k_x - \frac{eA_x}{c}}{m} \tag{10.6}$$

We need to define the quantum velocity operator and it also gives a similar form.

$$\frac{\hbar}{i}\dot{x} = [H, x] \tag{10.7}$$

$$= \left(\frac{(p_x + \hbar k_x - \frac{eA_x}{c}}{m}\right)[p_x, x] \tag{10.8}$$

By using the commutation relations $[A^2, B] = 2A[A, B]$ and $[p_x, x] = \frac{-\hbar}{i}$ we get the similar form of velocity.

$$\dot{x} = v_x = \frac{(p_x + \hbar k_x - \frac{eA_x}{c}}{m} \tag{10.9}$$

Now we need to calculate $\langle\beta|v_x|\alpha\rangle$.

$$\langle\beta|v_x|\alpha\rangle = \langle\beta|\frac{1}{\hbar}\frac{\partial H}{\partial k_x}|\alpha\rangle \tag{10.10}$$

We would use the following manipulation:

$$\langle\beta|H|\alpha\rangle = 0 \tag{10.11}$$

$$\frac{\partial}{\partial k_x}\langle\beta|H|\alpha\rangle = 0 \tag{10.12}$$

$$\langle\frac{\partial\beta}{\partial k_x}|H|\alpha\rangle + \langle\beta|\frac{\partial H}{\partial k_x}|\alpha\rangle + \langle\beta|H|\frac{\partial\alpha}{\partial k_x}\rangle = 0 \tag{10.13}$$

and similarly

$$\langle\beta|\alpha\rangle = 0 \tag{10.14}$$

$$\frac{\partial}{\partial k_x}\langle\beta|\alpha\rangle = 0 \tag{10.15}$$

$$\langle\frac{\partial\beta}{\partial k_x}|\alpha\rangle + \langle\beta|\frac{\partial\alpha}{\partial k_x}\rangle = 0 \tag{10.16}$$

And also $H|\alpha\rangle = \varepsilon_\alpha H|\alpha\rangle$ and $H|\beta\rangle = \varepsilon_\beta H|\beta\rangle$.

So therefore,

$$\langle\beta|v_x|\alpha\rangle = -\langle\frac{\partial\beta}{\partial k_x}|H|\alpha\rangle - \langle\beta|H|\frac{\partial\alpha}{\partial k_x}\rangle \tag{10.17}$$

$$= -\varepsilon_\alpha\langle\frac{\partial\beta}{\partial k_x}|\alpha\rangle - \varepsilon_\beta\langle\beta|\frac{\partial\alpha}{\partial k_x}\rangle \tag{10.18}$$

$$= (\varepsilon_\alpha - \varepsilon_\beta)\langle\beta|\frac{\partial\alpha}{\partial k_x}\rangle \tag{10.19}$$

Now we plug these in to our favourite Kubo's formula:

$$\sigma_{xy} = -\frac{e^2\hbar}{iA} \sum_{\varepsilon_\alpha < \varepsilon_F < \varepsilon_\beta} \frac{\langle\alpha|v_x|\beta\rangle\langle\beta|v_y|\alpha\rangle - \langle\alpha|v_y|\beta\rangle\langle\beta|v_x|\alpha\rangle}{(\varepsilon_\alpha - \varepsilon_\beta)^2} \tag{10.20}$$

$$= -\frac{e^2\hbar}{iA} \sum \langle \frac{1}{\hbar}\frac{\partial\alpha}{\partial k_x}|\beta\rangle\langle\beta|\frac{1}{\hbar}\frac{\partial\alpha}{\partial k_y}\rangle - \langle\frac{1}{\hbar}\frac{\partial\alpha}{\partial k_y}|\beta\rangle\langle\beta|\frac{1}{\hbar}\frac{\partial\alpha}{\partial k_x}\rangle \tag{10.21}$$

Here α represents the occupied levels and β represents unoccupied levels. We use the identity of sum over all states:

$$I = \sum |\alpha\rangle\langle\alpha| + \sum |\beta\rangle\langle\beta| \tag{10.22}$$

Now we plug this in the Kubo representation above

$$\sigma_{xy} = -\frac{e^2}{iA\hbar} \sum \langle \frac{\partial\alpha}{\partial k_x}|(1 - \sum_{\alpha'} |\alpha'\rangle\langle\alpha'|)\frac{\partial\alpha}{\partial k_y}\rangle - \langle\frac{\partial\alpha}{\partial k_y}|(1 - \sum_{\alpha'} |\alpha'\rangle\langle\alpha'|)|\frac{\partial\alpha}{\partial k_x}\rangle \tag{10.23}$$

Claim:

$$\sum_{\alpha\alpha'} \left[\langle\frac{\partial\alpha}{\partial k_x}|\alpha'\rangle\langle\alpha'|\frac{\partial\alpha}{\partial k_y}\rangle - \langle\frac{\partial\alpha}{\partial k_y}|\alpha'\rangle\langle\alpha'|\frac{\partial\alpha}{\partial k_x}\rangle \right] = 0 \tag{10.24}$$

We can shoe this by using the relation in equation() and then relabeling the terms. So now we have only this term left:

$$\sigma_{xy} = -\frac{e^2}{iA\hbar} \sum_\alpha \left[\langle\frac{\partial\alpha}{\partial k_x}|\frac{\partial\alpha}{\partial k_y}\rangle - \langle\frac{\partial\alpha}{\partial k_y}|\frac{\partial\alpha}{\partial k_x}\rangle \right] \tag{10.25}$$

As $\langle x|\alpha\rangle = u_k^\alpha(x)$

$$\sigma_{xy} = -\frac{e^2}{iA\hbar} \sum_\alpha \int d^2x \left[\frac{\partial u_k^*}{\partial k_x}\frac{\partial u_k}{\partial k_y} - \frac{\partial u_k^*}{\partial k_y}\frac{\partial u_k}{\partial k_x} \right] \tag{10.26}$$

The sum is over all states and we can replace it with an integral, where it is integrated over the whole magnetic Brillouin Zone (B.Z)

$$\sum_\alpha = \frac{A}{(2\pi)^2} \int_{B.Z} d^2k \tag{10.27}$$

$$\sigma_{xy} = -\frac{e^2}{h2\pi i}\int d^2k \int d^2x \left[\frac{\partial u_k^*}{\partial k_x}\frac{\partial u_k}{\partial k_y} - \frac{\partial u_k^*}{\partial k_y}\frac{\partial u_k}{\partial k_x}\right] \tag{10.28}$$

We now define a vector potential like quantity $\vec{A}$

$$\vec{A} = \int d^2x \ u_k^* \ \vec{\nabla}_k \ u_k \tag{10.29}$$

$$(\vec{\nabla}_k \times \vec{A})_3 = \frac{\partial A_y}{\partial k_x} - \frac{\partial A_x}{\partial k_y} \tag{10.30}$$

We write equation in a form like:

$$\sigma_{xy} = -\frac{e^2}{h2\pi i}\int d^2k \int d^2x \left[\frac{\partial}{\partial k_x}(u_k^*\frac{\partial}{\partial k_y}u_k) - \frac{\partial}{\partial k_y}(u_k^*\frac{\partial}{\partial k_x}u_k)\right] \tag{10.31}$$

Plug in from equation () and we get an alternate form of the Kubo's conductance, the third component of the vector potential like term which means that it points in the z direction.

$$\sigma_{xy} = -\frac{e^2}{h}\frac{1}{2\pi i}\int d^2k(\vec{\nabla}_k \times \vec{A})_3 \tag{10.32}$$

The integration is over the whole magnetic brillouin zone; $0 \leq k_1 \leq \frac{2\pi}{qa}, 0 \leq k_2 \leq \frac{2\pi}{b}$. Note that this is a Torus in k-space, which means that $k_1 = 0$ and $\frac{2\pi}{qa}$ are the same points, similarly $k_2 = 0$ and $\frac{2\pi}{b}$ are the same too.

Since the Torus does not have a boundary, the application of stokes theorem to equation would give $\sigma_{xy}^{\alpha} = 0$ if $\vec{A}(k_1, k_2)$ is uniquely defined on the entire torus. A non-trivial $\vec{A}(k_1, k_2)$ can only be constructed when the global topology of the base space is non-contractible.

Now we would try to understand one such non-trivial topology. Suppose $u_{k_1k_2}(x, y)$ and $u_{k_1k_2}(x, y)e^{if(k_1k_2)}$ where $f(k_1k_2)$ is an arbitrary smooth function of k_1 and k_2

So we introduce transformations:

$$u'_{k_1k_2}(x, y) \to u_{k_1k_2}(x, y)e^{if(k_1k_2)} \tag{10.33}$$

$$\vec{A}'(k_1, k_2) = \vec{A}(k_1, k_2) + i\nabla_k f(k_1, k_2) \tag{10.34}$$

You can see that Eqs () and () remain invariant under this transformation.

Non trivial arises when there are zeros of $u_{k_1k_2}(x, y)$, for simplicity we consider the case where $u_{k_1k_2}(x, y)$ just vanishes at one point in the magnetic Brillouin zone. Anywhere where $u_{k_1k_2}(x, y) = 0$, there is an ambiguity, you could multiply different things and still get the same result. So that means f is not necessarily a continuous function.

See figure (), suppose $u_{k_1k_2}(x, y)$ vanishes at the centre, so we isolate that patch. Now we have two patches H_I and H_{II}, and at the boundary of these two patches we have a phase mismatch.

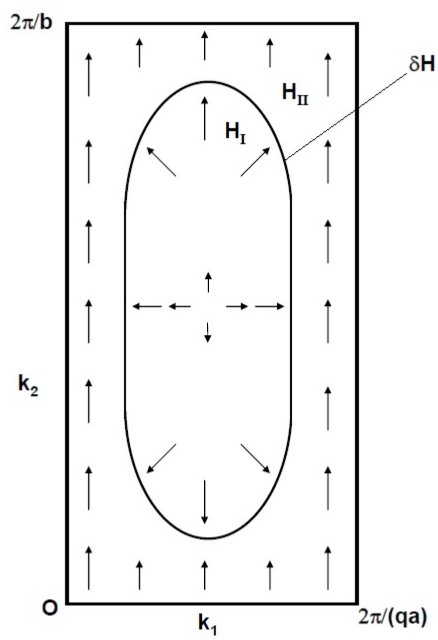

Figure 5: A magnetic unit cell in k space, it has a geometry of a Torus. The zero of U is in the center of the patch and we have isolated it

$$|u^{II}_{k_1k_2}\rangle = exp[i\chi(k_1, k_2)]|u^{I}_{k_1k_2}\rangle \tag{10.35}$$

where $\chi(k_1, k_2)$ is a smooth function on the boundary ∂H. Smooth vector fields $\vec{A_I}(k_1, k_2)$ and $\vec{A_{II}}(k_1, k_2)$ are defined on H_I and H_{II} respectively. The phase mismatch of the state vector induces the following relation between $\vec{A_I}(k_1, k_2)$ and $\vec{A_{II}}(k_1, k_2)$ on the boundary ∂H.

$$\vec{A_{II}}(k_1, k_2) = \vec{A_I}(k_1, k_2) + i\nabla_k \chi(k_1, k_2) \tag{10.36}$$

Now we can apply stokes theorem to H_I and H_{II} separately.

$$\sigma_{xy} = -\frac{e^2}{h}\frac{1}{2\pi i}\left\{\int_{H_I} d^2k(\vec{\nabla}_k \times \vec{A_I})_3 + \int_{H_{II}} d^2k(\vec{\nabla}_k \times \vec{A_{II}})_3\right\} \tag{10.37}$$

Now note that as it is a Torus, it is a closed surface when you take an integral over H_{II} it is the same as walking about H_I but in the other direction. You can visualise it by imagining two stickmen walking clockwise on a patch on the surface of a torus. They both would walk on the boundary of the same patch but in opposite directions. So the above equation then becomes:

$$\sigma_{xy} = -\frac{e^2}{h}\frac{1}{2\pi i}\int_{\partial H} dk.[\vec{A_I} - \vec{A_{II}}] \tag{10.38}$$

Using the relation between $\vec{A_I}$ and $\vec{A_{II}}$ from equation (), our conductivity becomes:

$$\sigma_{xy} = \frac{e^2}{h} n \tag{10.39}$$

where

$$n = \frac{1}{2\pi}\int_{\partial H} dk.\nabla_k \chi(k_1, k_2) \tag{10.40}$$

n must be an integer for each state vectors when we have a compete revolution about ∂H. We showed that n is an integer in the previous section, equation ().

We have shown how the conductance $\sigma_{xy} = \frac{e^2}{h} n$ is topologically protected.

11 Kubo Formula from Green's Theory

11.1 Introduction

The starting point is equation 9.5 of Fetter and Walecka. For now we will forget about spin.

$$iG(x,y) = \sum_{m=0}^{\infty} \left(\frac{-i}{\hbar}\right)^m \frac{1}{m!} \int_{-\infty}^{+\infty} dt_1 \cdots \int_{-\infty}^{+\infty} dt_m \langle \Phi_o | T[H(t_1)\cdots H(t_m)\psi(x)\psi^\dagger(y)] | \Phi_o \rangle \tag{11.1}$$

Here, Φ_o is the free ground state, $H(t_1)\cdots H(t_m)$ is the perturbing Hamiltonian in the interaction representation, $\Psi(x)\Psi^\dagger(y)$ are the fermion field operators in the interaction representation.

Crucial Point: Only linked diagrams are to be included. Although it is not obvious, the above holds even when the perturbing Hamiltonian is time dependent in the schrodinger equation.

We are interested with $H(t') = \frac{e^{iwt'}}{w} \int d^2x' \psi^\dagger(x')\pi\psi(x')$ where π will be proportional to $A.\nabla$. The $w \to 0$ limit will have to be taken for the dc case.

Before we go any further we will do a brief review of particles and holes.

11.2 Review of Particles and Holes

Let $u_i(x)$ be the complete set of eigenstates of H_o

$$H_o u_i = \varepsilon_i u_i \quad \textit{where } i \equiv k_x, k_y \textit{ band indices.} \tag{11.2}$$

Let $\{a_i, a_j\dagger\} = \delta_{ij}$. Then the second quantized Hamiltonian (only one body term) is:

$$H = \sum \varepsilon_\alpha a_\alpha^\dagger a_\alpha + \sum_{\alpha\beta} \langle \beta | H' | \alpha \rangle a_\beta^\dagger a_\alpha \tag{11.3}$$

This is correct because,

$$\langle r|H'|s\rangle = \langle 0|a_r H' a_s^\dagger|0\rangle \tag{11.4}$$

$$= \sum_{\alpha\beta} \langle\beta|H'|\alpha\langle 0|a_r a_\beta^\dagger H' a_\alpha a_s^\dagger|0\rangle \tag{11.5}$$

$$= \langle r|H'|s\rangle \tag{11.6}$$

Fields:

$$\varphi(x) = \sum_\alpha u_\alpha(x) a_\alpha \tag{11.7}$$

$$\varphi^\dagger(x) = \sum_\alpha u_\beta(x) a_\beta^\dagger \tag{11.8}$$

$$\{\varphi(x), \varphi^\dagger(x')\} = \sum_{\alpha\beta} u_\alpha(x) u_\beta(x') \{a_\alpha a_\beta\} \tag{11.9}$$

$$= \sum_\alpha u_\alpha(x) u_\alpha(x') \tag{11.10}$$

$$= \delta(x - x') \tag{11.11}$$

In second form,

$$H' = \int dx \varphi^\dagger(x) H'(x) \varphi(x) \tag{11.12}$$

This is true as,

$$H' = \int dx \varphi^\dagger(x) H'(x) \varphi(x) \tag{11.13}$$

$$= \sum_{\alpha\beta} \int dx \varphi_\beta^*(x) H'(x) \varphi_\alpha(x) a_\beta^\dagger a_\alpha \tag{11.14}$$

$$= \sum_{\alpha\beta} \langle\beta|H'|\alpha\rangle a_\beta^\dagger a_\alpha \tag{11.15}$$

Now come to particles and holes.

Define:

$$a_\alpha = \begin{cases} b_\alpha & \text{if } \varepsilon_\alpha > \varepsilon_F \\ c_\alpha^\dagger & \text{if } \varepsilon_\alpha < \varepsilon_F \end{cases} \tag{11.16}$$

where, $\{b_\alpha, b_\beta\} = \delta_{\alpha\beta}$ and $\{c_\alpha, c_\beta\} = \delta_{\alpha\beta}$.

$$\varphi(x) = \sum_{\varepsilon_\alpha < \varepsilon_F} \varphi_\alpha(x) c_\alpha^\dagger + \sum_{\varepsilon_\beta > \varepsilon_F} \varphi_\beta(x) b_\beta^\dagger \{\varphi(x), \varphi^\dagger(y)\} = \delta(x-y) \tag{11.17}$$

In terms of particles and holes, H_o is:

$$H_o = \sum_\alpha \varepsilon_\alpha a_\alpha^\dagger a_\alpha + \sum_\beta \varepsilon_\beta a_\beta^\dagger a_\beta \tag{11.18}$$

$$= \sum_\alpha \varepsilon_\alpha c_\alpha c_\alpha^\dagger + \sum_\beta \varepsilon_\beta b_\beta^\dagger b_\beta \tag{11.19}$$

$$= -\sum_\alpha \varepsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_\beta \varepsilon_\beta b_\beta^\dagger b_\beta + \sum_\alpha \varepsilon_\alpha \tag{11.20}$$

from $i\hbar \dot{c_\alpha} = [c_\alpha, H_o] = -\varepsilon_\alpha c_\alpha$

$\therefore c_\alpha = c_\alpha(0) e^{i\varepsilon_\alpha t}$

So in the interaction representation:

$$\varphi(x,t) = \sum_\alpha e^{-i\varepsilon_\alpha t/\hbar} \varphi_\alpha(x) c_\alpha^\dagger + \sum_\beta e^{-i\varepsilon_\beta t/\hbar} \varphi_\beta(x) b_\alpha \tag{11.21}$$

$$\varphi^\dagger(x,t) = \sum_\alpha e^{i\varepsilon_\alpha t/\hbar} \varphi_\alpha^*(x) c_\alpha + \sum_\beta e^{i\varepsilon_\beta t/\hbar} \varphi_\beta^*(x) b_\alpha^\dagger \tag{11.22}$$

11.3 Zeroth order Green's function

Now we will take equation 1 and carefully workout the $m = 0$ term:

$$iG^o(xt, x't') = \langle \Phi_o | T[\varphi(xt)\varphi^\dagger(x't')] | \Phi_o \rangle \tag{11.23}$$

$$= \theta(t-t') \langle \Phi_o | \varphi(xt) \varphi^\dagger(x't') | \Phi_o \rangle - \theta(t'-t) \langle \Phi_o | \varphi^\dagger(x't') \varphi(xt) | \Phi_o \rangle \tag{11.24}$$

$$\tag{11.25}$$

Now $\varphi^\dagger(x't')|\Phi_o\rangle \sim c|\Phi_o\rangle + b^\dagger|\Phi_o\rangle \sim b^\dagger|\Phi_o\rangle$ as $c|\Phi_o\rangle = 0$.

Similarly $\varphi(xt)|\Phi_o\rangle \sim c^\dagger|\Phi_o\rangle + b|\Phi_o\rangle \sim c^\dagger|\Phi_o\rangle$.

$$iG^o(xt, x't') = \theta(t-t')\sum_\beta e^{-i\varepsilon_\beta(t-t')}\varphi_\beta(x)\varphi^*_\beta(x') - \theta(t'-t)\sum_\alpha e^{-i\varepsilon_\alpha(t-t')}\varphi_\alpha(x)\varphi^*_\alpha(x') \tag{11.26}$$

$$= \sum_\alpha \int \frac{dw}{2\pi i}\frac{e^{-i(w+\varepsilon_\alpha)(t-t')}}{w-i\eta}\varphi_\alpha(x)\varphi^*_\alpha(x') - \sum_\beta \int \frac{dw}{2\pi i}\frac{e^{-i(-w+\varepsilon_\beta)(t-t')}}{w+i\eta}\varphi_\beta(x)\varphi^*_\beta(x') \tag{11.27}$$

$$= \sum_\alpha \int \frac{dw}{2\pi i}\frac{e^{-iw(t-t')}}{\hbar w-\varepsilon_\alpha-i\eta}\varphi_\alpha(x)\varphi^*_\alpha(x') + \sum_\beta \int \frac{dw}{2\pi i}\frac{e^{-iw(t-t')}}{\hbar w-\varepsilon_\beta+i\eta}\varphi_\beta(x)\varphi^*_\beta(x') \tag{11.28}$$

where we have used the identity:

$$\theta(t-t') = -\int \frac{dw}{2\pi i}\frac{e^{-iw(t-t')}}{w+i\eta} \tag{11.29}$$

11.4 First order Green's function

$$iG^{(1)} = \frac{-i}{\hbar}\int_{-\infty}^{+\infty} dt' \langle\Phi_o|T[H(t')\varphi(xt)\varphi^\dagger(yt)]|\Phi_o\rangle \tag{11.30}$$

with

$$H(t') = \frac{e^{iwt'}}{w}\int d^2x'\varphi^\dagger(x')\Pi\varphi(x) \tag{11.31}$$

So we have

$$\langle\Phi_o|T[\varphi^\dagger(x')\Pi\varphi(x')\varphi(x)\varphi^\dagger(y)]|\Phi_o\rangle \tag{11.32}$$

By Wick's theorem this becomes

$$\langle\Phi_o|\varphi(x)\varphi^\dagger(x')\Pi\varphi(x')\varphi^\dagger(y)|\Phi_o\rangle \tag{11.33}$$

Therefore

$$iG^{(1)} = \frac{-i}{\hbar}\int dt'd^2x' iG^o(x,x')\Pi iG^o(x',y) \tag{11.34}$$

$$G^{(1)} = \frac{1}{\hbar}\int dt'd^2x' G^o(x,x')\Pi G^o(x',y) \tag{11.35}$$

So for solving we would use the following structure:

$$G^o \pi G^o \sim (G^+ + G^-)\Pi(G^+ + G^-) \tag{11.36}$$

where the superscript + represents the term in G^o with $\frac{1}{w-\varepsilon+i\eta}$ term and superscript $-$ represents $\frac{1}{w-\varepsilon-i\eta}$ term.

We first calculate $(G^+\pi G^-)$:

$$\frac{1}{\hbar}\int dt' d^2x' \sum_\beta \int \frac{dw_1}{2\pi i}\frac{e^{-iw_1(t_x-t')}}{w_1-\varepsilon_\beta+i\eta}\varphi_\beta(x)\varphi^*_\beta(x') * \frac{e^{iwt'}}{w}\Pi\sum_\alpha \int \frac{dw_2}{2\pi i}\frac{e^{-iw_2(t'-ty)}}{w_2-\varepsilon_\alpha-i\eta}\varphi_\alpha(x')\varphi^*_\alpha(y) \tag{11.37}$$

Now, $\int dt' e^{i(w_1+w-w_2)t'} = 2\pi\delta(w_1+w-w_2)$, so our equation becomes:

$$\frac{1}{i\hbar w}\sum_{\alpha\beta}\langle\beta|\Pi\alpha\rangle\varphi_\beta(x)\varphi^*_\alpha(y)\int \frac{dw_1}{2\pi i}\frac{e^{iw_1(t_y-t_x)}e^{iwt_y}}{(w_1-\varepsilon_\beta+i\eta)(w_2-\varepsilon_\alpha-i\eta)} \tag{11.38}$$

As

$$\int \frac{dw_1}{2\pi i}\frac{e^{iw_1(t_y-t_x)}}{(w_1-\varepsilon_\beta+i\eta)(w_2-\varepsilon_\alpha-i\eta)} \tag{11.39}$$

$$= \frac{e^{i(\varepsilon_\alpha-w)(t_y-t_x)}}{\varepsilon_\alpha-\varepsilon_\beta-w+i\eta} \tag{11.40}$$

So our equation simplifies to:

$$\frac{1}{i\hbar w}e^{iwt_y}\sum_{\alpha\beta}\frac{\langle\beta|\Pi|\alpha\rangle\varphi_\beta(x)\varphi^*_\alpha(y)}{\varepsilon_\alpha-\varepsilon_\beta-w+i\eta}e^{i(\varepsilon_\alpha-w)(t_y-t_x)} \tag{11.41}$$

We put $\eta = 0$ and $t_y = t_x$ as they are no longer needed. And finally we get:

$$(G^+\Pi G^-) = \frac{e^{iwt_y}}{i\hbar w}\sum_{\alpha\beta}\frac{\langle\beta|\Pi|\alpha\rangle\varphi_\beta(x)\varphi^*_\alpha(y)}{\varepsilon_\alpha-\varepsilon_\beta-w} \tag{11.42}$$

Similarly we calculate $(G^+\Pi G^+), (G^-\Pi G^-), (G^-\Pi G^+)$. Turns out that only $(G^-\Pi G^+)$ is non zero. So we add $(G^+\Pi G^-)$ and $(G^-\Pi G^+)$ and get:

$$(G^+ + G^-)\Pi(G^+ + G^-) = (G^+\Pi G^-) + (G^-\Pi G^+) \tag{11.43}$$

$$= \frac{e^{iwt_y}}{i\hbar w}\sum_{\alpha\beta}\left(\frac{\langle\beta|\Pi|\alpha\rangle\varphi_\beta(x)\varphi^*_\alpha(y)}{\varepsilon_\alpha - \varepsilon_\beta - w} + \frac{\langle\alpha|\Pi|\beta\rangle\varphi_\alpha(x)\varphi^*_\beta(y)}{\varepsilon_\alpha - \varepsilon_\beta + w}\right) \tag{11.44}$$

11.5 Extension to Kubo Formula

For any one body operator $J = \int d^3x J(x)$.

$$\langle J(x)\rangle = -i lim_{t'\to t^+} lim_{x'\to x} J(x)G(xt, x't') \tag{11.45}$$

Now recall that

$$H'(t) = -\frac{ie}{mw}E_j p_j e^{iwt} \tag{11.46}$$

$$= \frac{e^{iwt}}{w}\Pi \tag{11.47}$$

Therefore:

$$\langle ev_i\rangle = \frac{\langle\Phi|e\hat{v}_i|\Phi\rangle}{\langle\Phi|\Phi\rangle} \tag{11.48}$$

$$= \frac{ie^2}{\hbar w}\sum_{\alpha\beta}\left(\frac{\langle\alpha|\hat{v}_i|\beta\rangle\langle\beta|\hat{v}_j|\alpha\rangle}{\varepsilon_\alpha - \varepsilon_\beta - w} + \frac{\langle\alpha|\hat{v}_j|\beta\rangle\langle\beta|\hat{v}_i|\alpha\rangle}{\varepsilon_\alpha - \varepsilon_\beta + w}\right)E_j \tag{11.49}$$

and so,

$$\sigma_{ij} = \frac{ie^2}{\hbar w}\sum_{\alpha\beta}\left(\frac{\langle\alpha|\hat{v}_i|\beta\rangle\langle\beta|\hat{v}_j|\alpha\rangle}{\varepsilon_\alpha - \varepsilon_\beta - w} + \frac{\langle\alpha|\hat{v}_j|\beta\rangle\langle\beta|\hat{v}_i|\alpha\rangle}{\varepsilon_\alpha - \varepsilon_\beta + w}\right) \tag{11.50}$$

We can expand $\frac{1}{\varepsilon_\alpha - \varepsilon_\beta - w} = \frac{1}{\varepsilon_\alpha - \varepsilon_\beta} + \frac{w}{(\varepsilon_\alpha - \varepsilon_\beta)^2} + \cdots$.

The first term vanishes as we showed earlier and we get

$$\sigma_{ij} = ie^2\hbar\sum_{\alpha\beta}\frac{\langle\alpha|\hat{v}_i|\beta\rangle\langle\beta|\hat{v}_j|\alpha\rangle - \langle\alpha|\hat{v}_j|\beta\rangle\langle\beta|\hat{v}_i|\alpha\rangle}{(\varepsilon_\alpha - \varepsilon_\beta)^2} \tag{11.51}$$

So we have reproduced the Kubo formula from Green's function theory.

12 Kubo and Beyond, spinning disk with magnetic field

$$\langle v_i \rangle = iw \sum_{\alpha\beta_1\beta_2} \frac{\langle\alpha|\hat{v}_i|\beta_1\rangle\langle\beta_1|\hat{v}_j|\beta_2\rangle - \langle\alpha|\hat{v}_j|\beta_1\rangle\langle\beta_1|\hat{v}_i|\beta_2\rangle}{(\varepsilon_\alpha - \varepsilon_{\beta_1})^2(\varepsilon_\alpha - \varepsilon_{\beta_2})} \langle\beta_2|\Pi_L|\alpha\rangle \tag{12.1}$$

$$- \frac{iw}{2} \sum_{\alpha\beta_1\beta_2} \frac{\langle\beta_2|\hat{v}_i|\alpha\rangle\langle\alpha|\hat{v}_j|\beta_1\rangle - \langle\beta_2|\hat{v}_j|\alpha\rangle\langle\alpha|\hat{v}_i|\beta_1\rangle}{(\varepsilon_\alpha - \varepsilon_{\beta_1})(\varepsilon_\alpha - \varepsilon_{\beta_2})} \langle\beta_1|\Pi_L|\beta_2\rangle * \left(\frac{1}{\varepsilon_\alpha - \varepsilon_{\beta_1}} + \frac{1}{\varepsilon_\alpha - \varepsilon_{\beta_2}}\right) \tag{12.2}$$

$$- \frac{iw}{2} \sum_{\alpha_1\alpha_2\beta} \frac{\langle\alpha_2|\hat{v}_i|\beta\rangle\langle\beta|\hat{v}_j|\alpha_1\rangle - \langle\alpha_2|\hat{v}_j|\beta\rangle\langle\beta|\hat{v}_i|\alpha_1\rangle}{(\varepsilon_{\alpha_1} - \varepsilon_\beta)(\varepsilon_{\alpha_2} - \varepsilon_\beta)} \langle\alpha_1|\Pi_L|\alpha_2\rangle * \left(\frac{1}{\varepsilon_{\alpha_1} - \varepsilon_\beta} + \frac{1}{\varepsilon_{\alpha_2} - \varepsilon_\beta}\right) \tag{12.3}$$

$$+ iw \sum_{\alpha_1\alpha_2\beta} \frac{\langle\alpha_2|\hat{v}_i|\alpha_1\rangle\langle\alpha_1|\hat{v}_j|\beta\rangle - \langle\alpha_2|\hat{v}_j|\alpha_1\rangle\langle\alpha_1|\hat{v}_i|\beta\rangle}{(\varepsilon_{\alpha_1} - \varepsilon_\beta)^2(\varepsilon_{\alpha_2} - \varepsilon_\beta)} \langle\beta|\Pi_L|\alpha_2\rangle \tag{12.4}$$

13 Future Plans

14 Appendix

14.1 Appendix A: Aharanov Bohm Effect

14.1.1 Scalar and Vector fields

Some time ago people believed that the vector and scalar potentials are extraneous and are not necessary, we only use them as crutches to make representations cleaner and easier to interpret but if we do the physics carefully there would be no need to invoke the concepts of vector and scalar potentials. But ofcourse they were wrong, as we will see later the importance of the vector potential.

We all are familiar with the four maxwell equations:

$$\nabla \cdot \vec{E} = 4\pi\rho \tag{14.1}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{14.2}$$

$$\nabla \cdot \vec{B} = 0 \tag{14.3}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{14.4}$$

From equation (3) we can define

as $\nabla \cdot (\nabla \times A) = 0$; divergence of a curl is always zero and A is a vector field. Now we substitute equation (5) into the maxwell equation (2) and we get:

$$\nabla \times E = -\frac{1}{c}\ \nabla \times \frac{\partial A}{\partial t}$$

$$\nabla \times \underbrace{(E + \frac{1}{c}\ \frac{\partial A}{\partial t})} = 0$$

and we can define

$$(E + \frac{1}{c}\ \frac{\partial A}{\partial t}) = -\nabla\phi \tag{14.5}$$

where $\nabla\phi$ is a gradient of a scalar field, by using the relation:

$$\nabla \times (\nabla\phi) = 0$$

rearranging eq(6) we get

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \tag{14.6}$$

where ϕ is a scalar field and $\vec{A}$ is a vector field

14.1.2 Guage freedom

Now that we have defined our E field and B field in terms of field potentials (eq(5) and eq(7)), we can apply some some guage transformations to these fields and see that they are guage invariant for some arbitrary guages and when substituted, give back the same E and B fields back. We set the following Guage Transformations:

$$\vec{A} \rightarrow \vec{A} + \nabla\chi \tag{14.7}$$

$$\phi \rightarrow \phi - \frac{1}{c}\frac{\partial\chi}{\partial t} \tag{14.8}$$

where χ is a scalar field.

Now substitute these guage transformations into equations (5) and (7) and you will see that they don't change:

$$\vec{E} \rightarrow -\nabla(\phi - \frac{1}{c}\frac{\partial\chi}{\partial t}) - \frac{1}{c}\frac{\partial}{\partial t}(\vec{A} + \nabla\chi) \tag{14.9}$$

$$\vec{B} \rightarrow \nabla \times (\vec{A} + \nabla\chi) \tag{14.10}$$

Further, we can see from equation (12) that the vector and scalar fields do not exert any force.

$$\vec{F} = e\vec{E} + \frac{e}{c}\vec{v} \times \vec{B} \tag{14.11}$$

14.1.3 Feynman's Path Integral Formulation

This is just a brief overview, a more intuitive and nicer derivation is in chapter I.2 of the book 'Quantum Field Theory in a nutshell' by A. Zee.

As in a double slit experiment, the amplitude of the particle is given by the superposition principle and is the sum of the amplitude for the particle to go from each hole. Similarly Feynman showed that to find the amplitude for the particle to propogate from the source to the detector is the sum of all possible paths.

So let $A_{i\rightarrow f}$ be the amplitude of going from $\mid i\rangle$ to $\mid f\rangle$ then $\mathcal{A}_i$ is the amplitude of each path and we sum them :

$$A_{i \to f} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \dots \tag{14.12}$$

$$= e^{\frac{i}{\hbar} S_1} + e^{\frac{i}{\hbar} S_2} + e^{\frac{i}{\hbar} S_3} + \dots \tag{14.13}$$

S_i refers to the action of the path and is given by:

$$S = \int dt \; L(x, \dot{x}) \tag{14.14}$$

where L is the lagrangian

$$L(x, \dot{x}) = \frac{1}{2} m v^2 + \frac{e}{c} \vec{v} \cdot \vec{A} \tag{14.15}$$

So in general the amplitude is given by :

$$\boxed{A_{i \to f} = \int Dx(t) \; e^{\frac{i}{\hbar} S}} \tag{14.16}$$

A classical particle would take the path with smallest action, but in Quantum Mechanics the particle will take all possible paths and the different actions give the amplitudes of the probability of different paths.

For a double slit experiment, if the individual amplitudes through the slits is $\mathcal{A}_1$ and $\mathcal{A}_2$ than the total amplitude is given by:

$$|A|^2 = |\mathcal{A}_1 + \mathcal{A}_2|^2$$
$$= |A_1|^2 + |A_2|^2 + A_1^* A_2 + A_2^* A_1$$

14.1.4 Aharanov Bohm Effect

This is a non intuitive and kind of a bizarre effect, so we take our favorite double slit experiment and place a solenoid towards the right of the slits and between them as shown in the figure. The solenoid can be thought of as coming out of the plane of the paper and has zero field outside and a magnetic field B inside.

From equations (14) and (15) we can write the amplitude of going through path 1 (slit 1) in terms of velocity $\vec{v}$ and vector potential $\vec{A}$ as:

$$\mathcal{A}_1 = A_1 \; e^{i \frac{e}{\hbar c} \int dt \; \vec{v} \cdot \vec{A}} \tag{14.17}$$

Using $\vec{v} = \frac{d\vec{x}}{dt}$ we get

$$\mathcal{A}_1 = A_1 \; e^{i \frac{e}{\hbar c} \int_1 d\vec{x} \cdot \vec{A}} \tag{14.18}$$

We can sum the two amplitudes going from each of the two slits

$$\mathcal{A}_1 + \mathcal{A}_2 = A_1 \; e^{i \frac{e}{\hbar c} \int_1 d\vec{x} \cdot \vec{A}} + A_2 \; e^{i \frac{e}{\hbar c} \int_2 d\vec{x} \cdot \vec{A}} \tag{14.19}$$

$$= e^{i \frac{e}{\hbar c} \int_1 d\vec{x} \cdot \vec{A}} \; [\; A_1 + A_2 \; e^{-i \frac{e}{\hbar c} \int_1 d\vec{x} \cdot \vec{A}} \cdot e^{i \frac{e}{\hbar c} \int_2 d\vec{x} \cdot \vec{A}} \;] \tag{14.20}$$

We can interpret the negative sign in the exponential above as a change in direction and now it looks more like a closed loop as shown in the figure.

$$e^{-i \frac{e}{\hbar c} \int_1 d\vec{x} \cdot \vec{A}} \cdot e^{i \frac{e}{\hbar c} \int_2 d\vec{x} \cdot \vec{A}} = e^{i \frac{e}{\hbar c} \oint d\vec{x} \cdot \vec{A}} \tag{14.21}$$

Now we all are familiar with Stokes Theorem:

$$\oint d\vec{x} \cdot \vec{A} = \int dS \cdot (\nabla \times \vec{A}) \tag{14.22}$$

$$= \int dS \cdot \vec{B} \tag{14.23}$$

$$= \Phi_B \tag{14.24}$$

And we use the relation $\nabla \times \vec{A} = \vec{B}$ and get Φ_B which is the flux of the Magnetic field $\vec{B}$.

Now equation (21) becomes:

$$\mathcal{A}_1 + \mathcal{A}_2 = e^{i \frac{e}{\hbar c}} \; [\; A_1 + e^{\frac{ie}{\hbar c} \Phi_B} \; A_2 \;] \tag{14.25}$$

where $e^{\frac{ie}{\hbar c} \Phi_B}$ is a phase factor. Now we can take the modulus of the sum of amplitudes as we did before and we get

$$|\mathcal{A}_1 + \mathcal{A}_2|^2 = |A_1|^2 + |A_2|^2 + e^{\frac{ie}{\hbar c} \Phi_B} \; A_1 A_2^* + e^{\frac{-ie}{\hbar c} \Phi_B} \; A_1^* A_2 \tag{14.26}$$

Now this shows that there would be a fluctuation depending on the $\vec{B}$ field. This is very profound and somewhat spooky too as we don't expect there to be any dependence on $\vec{B}$ field, the particle goes around the solenoid and never passes it and so it never experiences a force from the $\vec{B}$ field so intuitively there must be no fluctuation depending on the B field but yet there is. The only thing the particle comes in contact with is the vector potential $\vec{A}$ which pervades everywhere and so we can safely say that vector potentials are important and essential and not just crutches for representation.

Now we go define our vector potential in cylindrical coordinates:

$$\vec{A} = (A_z, A_\rho, A_\phi)$$

$$(\nabla \times \vec{A})_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}$$

We are only considering the vector potential in the $\hat{z}$ direction. We take $A_\rho = 0$ and so find a suitable A_ϕ for our case.

$$A_\phi = \begin{cases} \frac{1}{2}\vec{B}\rho & r < a \\ \\ \frac{\vec{B}a^2}{2\rho} & r > a \end{cases} \tag{14.27}$$

we set this value for $r > a$ so both potentials are equal at $r = 0$. We can also write this terms of the magnetic flux:

$$A_\phi = \begin{cases} \frac{1}{2}\vec{B}\rho & r < a \\ \\ \frac{\Phi_B}{2\pi\rho} & r > a \end{cases} \tag{14.28}$$

Further, $\vec{A} = \nabla\chi$ where $\chi = \frac{\Phi_B}{2\pi\rho}\phi$. χ is no longer single valued, because we don't know how many times it has gone around.

14.1.5 Energies

The hamiltonian is:

$$\vec{H} = \frac{\vec{p}^2}{2m} = \frac{1}{2m}(\frac{\hbar}{i}\vec{\nabla})^2 \tag{14.29}$$

By hamiltonian mechanics

$$\vec{p} \to \vec{p} - \frac{e}{c}\vec{A} \tag{14.30}$$

$$\vec{H} \to \frac{(\frac{\hbar}{i}\vec{\nabla})^2 - -\frac{e}{c}\vec{A}}{2m} \tag{14.31}$$

In polar coordinates we have $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}$, $\vec{A} = \hat{r}A_r + \hat{\theta}\frac{1}{r}A_\theta$ in which we ignore the $\frac{\partial}{\partial r} and A_r$ terms.

Now we solve the schrodinger equation using the above and take $r = 1$ to make our lives easier.

$$\frac{1}{2m}(\frac{\hbar}{i}\frac{\partial}{\partial\theta} - \frac{e}{c}A_\theta)^2 \ \Psi(\theta) = E \ \Psi(\theta) \tag{14.32}$$

$$\frac{1}{2m}(\frac{\hbar}{i}\frac{\partial}{\partial\theta} - \frac{e}{2\pi c}\Phi_B)^2 \ \Psi(\theta) = E \ \Psi(\theta) \tag{14.33}$$

Take $\Psi(\theta) = e^{i\alpha\theta}$ and then

$$\frac{\hbar}{i}\frac{\partial}{\partial\theta} \ e^{i\alpha\theta} = \hbar\alpha e^{i\theta} \tag{14.34}$$

where $\theta \to \theta + 2\pi$ and α/n have integer values.

And so we get our energy:

$$\boxed{E_n = \frac{1}{2m}(\hbar n - \frac{e}{2\pi c}\Phi_B)^2} \tag{14.35}$$

15 Bibliography